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An approach to constructing models of multiphase elastic porous media☆

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Abstract

A novel approach to describing the behaviour of multiphase elastic porous media is proposed. The average values of the physical quantities needed to describe the motions of porous media are formulated using an integral relation. The validity of this relation is taken as the fundamental hypothesis. The integral definition of the average values enables integral relations to be devised for the average values from the integral laws of conservation of mass, momentum and energy and the increase in entropy. Along with the average values, the integral relations contain new variables that can be identified with generalized thermodynamic forces, which can be used to take into account the phase interaction in a porous medium. The integral relations are used to derive differential equations for the rate of entropy change and Gibbs relations for a porous medium as a basis for obtaining the constitutive relations. Relationships between the thermomechanical parameters of the model are established from the Gibbs relations under additional assumptions. The equation for the rate of entropy change can be used to establish relations between the generalized thermodynamic forces and fluxes. A complete system of differential equations in the defining parameters, which describes the motion of multiphase elastic porous media, is finally obtained.

The mathematical difficulties that arise in describing the behaviour of a porous medium are associated mainly with its complicated internal structure. In real porous media, the pore space is a dense network of pores of diverse shape that are joined to one another by narrow slits and capillary channels, whose relative volume is considerably less than the relative volume of the pores. For this reason, all physical quantities fluctuate rapidly, i.e., the scale on which they undergo a noticeable change is much smaller than the characteristic dimensions of the bodies. The abrupt variation of the physical quantities is associated with another difficulty, viz., the large gradients of these quantities.

These difficulties can be avoided by averaging all the physical quantities (the density, velocity, stresses and internal energy) and deriving a system of averaged equations for them.[1,2] Smoothly varying fields of all the physical quantities, which are defined at all points of the region occupied by the porous medium, are thus obtained. The motion of a continuous medium in regions occupied by different phases is described in the general case by different systems of differential equations. The continuity conditions for the mass, momentum and energy fluxes must be satisfied on the phase boundaries. When the equations of the laws of conservation of mass, momentum and energy are averaged, equations of motion can be obtained for the averaged quantities throughout the region occupied by the porous medium. When the relations on the phase boundaries are averaged, they generate new quantities, viz., distributed phase-interaction forces

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and distributed sources of energy and heat, due to the interaction between the phases. Averaging does not enable the relationship between these quantities and the average values of the density, velocity, energy, etc to be established. The relationship between them is obtained using additional assumptions.

In solving many problems of practical importance, the solid phase can be regarded as a riged, non-deformable body. However, in some problems, for example, the wave propagation problem, the compressibility of the solid phase and the associated mutual influence of the properties of the solid and fluid phases on the motion of the porous medium must be taken into account.

When the behaviour of multiphase porous media is studied, the surface phenomena on the phase boundaries should be taken into account. This raises the question of how these phenomena can be taken into account on phase boundaries using distributed quantities. It is natural to assume that consideration of the surface tension forces should lead to certain relations between the average pressures of the fluid phases. Such an assumption was first made for a porous solid by Leverett,[3] who proposed a formula that relates the difference between the average pressures to the saturation of one of the phases. An extension of Leverett's law to the case of three fluids was given in Ref. 4.

This problem was solved most thoroughly by constructing a model of a three-phase elastic porous medium and a model of the simpler special case, viz., a two-phase porous medium.[2] However, several postulates that form the basis of these models were formulated specifically for a three-phase medium, and for this reason, the postulates introduced cannot be extended even formally to the case of an arbitrary number of phases. This applies primarily to the modelling of surface phenomena. Several important assumptions that were made in Ref. 2 were not supported by physical or other arguments in favour of their validity. Although such assumptions are not directly related to the fact that a porous medium is a three-phase medium, the missing foundation must be found before extending them to any multiphase medium.

Thus, the problem of constructing a model of the behaviour of multiphase elastic media cannot be considered to be completely solved. An approach that can be used to construct such a model for any number of fluid phases in the absence of phase transitions and chemical reactions is proposed below. The local definitions of the averaged quantities are not used here in constructing the model of a porous medium, and no dependence of the result on the order in which the differentiation and the taking of the average are carried out is found. We start out by defining averaged quantities using postulated integral relations for any region in a porous medium, whose validity is taken as a hypothesis.

1. Basic concepts and definitions

We will assume that a porous medium consists of one solid phase and n fluid phases. We will introduce several concepts and the notation associated with them. We take an arbitrary connected region Ω in a porous medium. We will represent Ω in the form of the union of two regions: $\Omega = \Omega\{0\} \cup \Omega\{f\}$. Here $\Omega\{0\}$ is the region occupied by the solid phase, and $\Omega\{f\}$ is the pore space. We will represent the pore space in the form of the union of n regions

$$\Omega\{f\} = \Omega\{1\} \cup \Omega\{2\} \cup \ldots \cup \Omega\{n\}$$

Here $\Omega\{\alpha\}$ is the region occupied by the α phase.

We will use $\partial\Omega\{\alpha\}$ to denote the boundary of the region $\Omega\{\alpha\}$, $\mathbf{n}_\alpha$ is the outward normal to $\partial\Omega\{\alpha\}$, $\Gamma\{\alpha\} \in \partial\Omega\{\alpha\}$ is the boundary between the α phase and the other phases of the porous medium, $\Gamma\{\alpha\beta\}$ is the boundary between the α and β phases, $\alpha \neq \beta$, $\Gamma\{\alpha\Omega\} = \partial\Omega\{\alpha\} \cap \partial\Omega$. We will assume that the normal to the boundaries $\Gamma\{\alpha\}$ and $\Gamma\{\alpha\beta\}$ coincides with $\mathbf{n}_\alpha$ ($\Gamma\{\alpha\beta\} \subset \Gamma\{\alpha\} \subset \partial\Omega\{\alpha\}$). If the boundary between the α and β phases is regarded as the subset $\partial\Omega\{\beta\}$, it can be denoted by $\Gamma\{\alpha\beta\}$ and has the normal $\mathbf{n}_\beta$. Obviously, $\mathbf{n}_\alpha = -\mathbf{n}_\beta$.

These definitions lead to obvious relations, which will be needed later to describe the averaging procedure:

$$\partial\Omega\{\alpha\} = \Gamma\{\alpha\} \cup \Gamma\{\alpha\Omega\}, \quad \Gamma\{\alpha\} = \bigcup_{\beta=0}^{n} \Gamma\{\alpha\beta\}, \quad \beta \neq \alpha;$$

$$\bigcup_{\alpha=0}^{n} \Gamma\{\alpha\} = \bigcup_{\alpha=0}^{n}\left(\bigcup_{\beta=0}^{n} \Gamma\{\alpha\beta\}\right) = \bigcup_{\alpha\neq\beta} \Gamma\{\alpha\beta\}, \quad \alpha \neq \beta$$

The modelling of a porous medium often starts out from the following simplifying representation. It is assumed that the fields of the physical quantities can be expanded into a sum of two components, viz., a smoothly varying quantity and a rapidly oscillating quantity, whose oscillations have an amplitude that is considerably less than the amplitude of the oscillations of the smoothly varying part.[5] The concept of an elementary neighbourhood $\delta v = \delta v(x)$ of each point x of the porous medium is introduced as part of these representations. Such a neighbourhood should be understood to be a sphere, whose centre is at the point x and whose radius is much greater than the characteristic distance between capillaries and, at the same time, is less than the characteristic distance over which significant changes in the physical fields occur.[1]

In the general case, the distribution of the capillary channels and pores is non-uniform and is characterized at each point x of the porous medium by the porosity $m = m(x, t)$, i.e., by the relative volume of the pore space of the elementary neighbourhood of the point $x \in \delta v$. Apart from the porosity, the important characteristics of a porous medium include the relative volumes of the phases m^α and the saturation of the fluid phases θ_α:

$$m^\alpha = |\delta v\{\alpha\}|/|\delta v|, \quad \alpha = 0, 1, \dots, n; \quad \theta_\alpha = m^\alpha/m, \quad \alpha = 1, 2, \dots, n$$

where $|\delta v\{\alpha\}|$ and $|\delta v|$ are the volumes of the regions $\delta v\{\alpha\}$ and $\delta v(\delta v\{\alpha\} \subset \delta v)$, and $\delta v\{\alpha\}$ is the part of the elementary neighbourhood δv occupied by the α phase. The neighbourhood $\delta v\{\alpha\}$ clearly depends on the point x. However, to avoid complicating the notation, we shall omit the variable x. It is easy to see that m^α and θ_α satisfy the relations

$$m^0 + m^1 + \dots + m^n = 1, \quad \theta_1 + \theta_2 + \dots + \theta_n = 1, \quad m^0 = 1 - m$$

Along with the relative volumes of the phases, we can introduce the so-called translucencies of the phases,[2] which are defined in the following manner. We draw an arbitrary plane Π_n with the normal $\mathbf{n}$ through a point $x \in \delta v$. We consider the cross sections of the neighbourhoods $\delta v\{\alpha\}$ and δv formed by this plane

$$\delta S = \delta v \cap \Pi_n, \quad \delta S\{\alpha\} = \delta v\{\alpha\} \cap \Pi_n$$

and we introduce the translucencies m_S^α, defined by the formula

$$m_S^\alpha = |\delta S\{\alpha\}|/|\delta S|, \quad \alpha = 0, 1, \dots, n$$

Here $|\delta S\{\alpha\}|$ and $|\delta S|$ are the areas of the cross sections $\delta S\{\alpha\}$ and δS, respectively, In the general case, m_S^α clearly depends on the orientation of the plane Π_n, and thus $m^\alpha \neq m_S^\alpha$. The conditions (regularity conditions) under which the translucencies depend only on the point in the porous medium will be indicated later on. The equalities $m^\alpha = m_S^\alpha$ hold under these conditions.

Now let $g = g(x, t)$ be some physical quantity that is assigned in the region Ω. The average value $g^\alpha(x, t)$ of g at a point $x \in \delta v\{\alpha\} \subset \Omega$ for the α phase is defined using the relations[1,2]

$$g^\alpha(x, t) = \frac{1}{|\delta v\{\alpha\}|} \iiint\limits_{\delta v\{\alpha\}} g\,dv, \quad \alpha = 0, 1, \dots, n \tag{1.1}$$

Here dv is a volume element. The average values $g_S^\alpha(x, t)$ of g over cross sections of the phases are introduced in a similar manner:

$$g_S^\alpha(x, t, \Pi_n) = \frac{1}{|\delta S\{\alpha\}|} \iiint\limits_{\delta S\{\alpha\}} g\,ds, \quad \alpha = 0, 1, \dots, n \tag{1.2}$$

Here ds is an element of area. In the general case, g_S^α depends both on the point $x \in \Omega$ and on the orientation of the plane Π_n, i.e., on the normal $\mathbf{n}$, and thus the average over the volume will not be equal to the average over the cross sections. Furthermore, averaged quantities defined in this manner depend on the shape and dimensions of the elementary neighbourhood δv. In order to eliminate these dependences, we must require satisfaction of the stability conditions[1]

$$\Delta_{\min} < \delta x < \Delta_{\max}, \quad d \ll \Delta_{\min} \ll \Delta_{\max} \ll L$$

Here δx is the range of linear dimensions, d is the size of the inhomogeneities, and L is the characteristic macroscopic dimension.

We shall assume that these conditions hold. We shall also assume that the following regularity conditions[1] also hold

$$\partial g^{\alpha}/\partial x \approx \partial g_S^{\alpha}/\partial x \leq g^{\circ}/L$$

Here g^{o} is the characteristic value of g or of a change in it. In this case, the following equalities hold[1]

$$g^{\alpha} = g_S^{\alpha}, \quad m^{\alpha} = m_S^{\alpha} \tag{1.3}$$

We introduce a quantity that plays an important role in the averaging procedure. We take the elementary neighbourhood of a point $x \in \delta v$ and the boundary between the α phase and the other phases in it. Since the neighbourhood δv contains a fairly large number of capillary channels, the area of the phase boundary $\Gamma\{\alpha\}$ greatly exceeds the area of the elementary neighbourhood or the area of any cross section of δv. In this case, based on geometric arguments, we can introduce the quantity

$$g_{\Gamma}^{\alpha} = \frac{1}{|\delta v|}\int\limits_{\Gamma\{\alpha\}} g ds \tag{1.4}$$

Departing from the conventional approach, we will define determine the averaged quantities differently. We will assume that the following two hypotheses hold.

1) For any physical quantity g assigned in a porous medium, the quantities g^{α} and g_{Γ}^{α} exist, which are such that the following relations hold for any region Ω in the porous medium

$$\iiint\limits_{\Omega\{\alpha\}} g dv = \iiint\limits_{\Omega} m^{\alpha} g^{\alpha} dv \tag{1.5}$$

$$\iint\limits_{\Gamma\{\alpha\}} g dS = \iiint\limits_{\Omega} g_{\Gamma}^{\alpha} dv \tag{1.6}$$

2) The following relation should hold for any surface S in a porous medium:

$$\iint\limits_{S\{\alpha\}} g ds = \iint\limits_{S} m^{\alpha} g^{\alpha} ds \tag{1.7}$$

We shall call the quantities g^{α} average values because the local definitions of the averaged quantities follow from integral relations (1.5)–(1.7) by to the mean-value theorem.

We shall henceforth call equality (1.6) the representability condition (this equality is sometimes named differently in the literature). It can be used to express surface integrals over the phase boundaries $\Gamma\{\alpha\}$ in terms of volume integrals and to introduce the quantities g_{Γ}^{α}. Depending on the physical meaning of g, these quantities are regarded below as phase-interaction forces, as the work of the phase-interaction forces or as the heat flux resulting from heat transfer between the phases.

We will introduce some notation: ρ is the density, $\mathbf{v}$ is the velocity vector, $\boldsymbol{\varepsilon}$ is the strain tensor, $\mathbf{e}(\mathbf{v})$ is the rate of deformation tensor, $\boldsymbol{\sigma}$ is the stress tensor, p is the spherical part of the stress tensor in the fluid phases ($p < 0$), $\boldsymbol{\tau}$ is the viscous stress tensor in the fluid phases (the deviator of the stress tensor in the solid phase), T is the temperature, e and s are the specific internal energy and entropy, respectively, $\mathbf{q}$ is the heat flux vector, $\mathbf{f}$ is volume force density, t is the time, $x = (x_1, x_2, x_3)$ denotes the spatial coordinates, $\nabla\mathbf{v}$ is the gradient of the velocity vector, $\mathbf{a} \cdot \mathbf{b}$ is the convolution of the two tensors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \otimes \mathbf{b}$ is the tensor product, and tensor $\mathbf{a}^t$ is the conjugate tensor $\boldsymbol{\alpha}$.

2. The differential equations for averaged quantities

As a starting point for deriving the system of equations which must be satisfied by the averaged quantities, we will use the integral relations corresponding to the laws of conservation of mass, momentum and energy and the second law of thermodynamics. A key role in the derivation of the equations is played by relations (1.5)–(1.7). Relation (1.5) is used to transform an integral over the region $\Omega\{\alpha\}$, which is occupied by the α phase, into integrals of the averaged quantities over Ω, and relation (1.7) is used to transform surface integrals over the cross section $S\{\alpha\} = \partial\Omega \cap \partial\Omega\{\alpha\}$ of the α phase into surface integrals over the boundary $\partial\Omega$. Finally, relation (1.6) can be used to transform surface integrals over the boundary $\Gamma\{\alpha\}$ between the α phase and adjacent phases into volume integrals of new, previously unknown quantities over the entire region Ω.

As we know, any conservation law can be written in the form of the differential relation

$$\partial a/\partial t + \nabla \cdot \mathbf{b} + c = 0 \tag{2.1}$$

Here a, $\mathbf{b}$ and c are certain physical quantities. In the class of piecewise-smooth functions, this equality is equivalent to the following integral relation

$$\frac{d}{dt}\iiint\limits_{G} a\,dv + \iint\limits_{\partial G}(\mathbf{b}\cdot\mathbf{n} - aw_n)ds + \iiint\limits_{G} c\,dv = 0 \tag{2.2}$$

as follows from the formula for differentiation of the integral over the variable region $G = G(t)$ with respect to a parameter (the time)[6] and the Gauss formula. Here G is any piecewise-smooth region in the porous medium, $\mathbf{n}$ is the outward normal to the boundary ∂G of the region G, $\mathbf{w}$ is the velocity vector of the boundary ∂G, and $w_n = \mathbf{w}\cdot\mathbf{n}$. The vector $\mathbf{w}$ is directed along the normal $\mathbf{n}$.

Well-known integral relations that correspond to the laws of conservation of mass, momentum and energy, as well as a relation that corresponds to the second law of thermodynamics, follow from (2.2). To obtain these relations, a, $\mathbf{b}$ and c must be replaced in (2.2) by the following quantities

$$a = \rho, \quad \mathbf{b} = -\rho\mathbf{v}, \quad c = 0$$

$$a = \rho\mathbf{v}, \quad \mathbf{b} = -\rho\mathbf{v}\otimes\mathbf{v} - \boldsymbol{\sigma}, \quad c = -\rho\mathbf{f}$$

$$a = \rho(e + \mathbf{v}^2/2), \quad \mathbf{b} = -a\mathbf{v} - \boldsymbol{\sigma}\cdot\mathbf{v} + \mathbf{q}, \quad c = 0$$

$$a = \rho s, \quad \mathbf{b} = -a\mathbf{v} + \mathbf{q}/T, \quad c = \mathbf{q}\cdot\nabla T/T^2$$

We now know how integral relations for averaged quantities can be obtained from (2.2).

We will take one of the regions $\Omega\{\alpha\} \subset \Omega$ occupied by the α phase as the region G. Note that the regions $\Omega\{\alpha\}$ are time-dependent. The boundary $\partial\Omega\{\alpha\}$ of the region $\Omega\{\alpha\}$ can be represented in the form $S\{\alpha\} \cup \Gamma\{\alpha\}$. The surface $S\{\alpha\}$ lies on the boundary $\partial\Omega$ of the region Ω, and $\Gamma\{\alpha\}$ is the boundary between the α phase and the adjacent phases of the porous medium. The surface $S\{\alpha\}$ is stationary, $w_n = 0$ on it, and $\Gamma\{\alpha\}$ is the contact boundary, on which the normal components of the velocity are identical on both sides of it and are equal to w_n, so that the equality $w_n = \mathbf{v}\cdot\mathbf{n} = v_n$ holds. Therefore, the surface integral over $\partial\Omega\{\alpha\}$ may be written in the form of the sum of two integrals:

$$\iint\limits_{\partial\Omega\{\alpha\}}(\mathbf{b}\cdot\mathbf{n} - aw_n)ds = \iint\limits_{S\{\alpha\}}\mathbf{b}\cdot\mathbf{n}ds + \iint\limits_{\Gamma\{\alpha\}}(\mathbf{b}\cdot\mathbf{n} - av_n)ds \tag{2.3}$$

We now use (1.5)–(1.7) and obtain integral relations for the averaged quantities from (2.2) and (2.3). For this purpose, we transform the integrals over the region $\Omega\{\alpha\}$ into integrals over Ω using (1.5), we transform the surface integral over $S\{\alpha\}$ into an integral over the boundary $\partial\Omega$ using (1.7), and we transform the integral over the boundary $\Gamma\{\alpha\}$ into a volume integral of the new, previously unknown quantity d^α over the region Ω using (1.6). We obtain the relation

$$\frac{d}{dt}\iiint\limits_{\Omega} m^\alpha a^\alpha dv + \iint\limits_{\partial\Omega} m^\alpha\mathbf{b}^\alpha\cdot\mathbf{n}ds + \iiint\limits_{\Omega} m^\alpha c^\alpha dv + \iiint\limits_{\Omega} d^\alpha dv = 0 \tag{2.4}$$

We transform the integral over the surface $\partial\Omega$ into a volume integral using the Gauss formula, and in the first term we reverse the order of integration and differentiation. This is possible because the region Ω is not time-dependent. Then we obtain an integral relation in the form of an equality of the integral over the region Ω to zero, from which it follows, by virtue of the arbitrariness of Ω, that the integrand is equal to zero, i.e., for the averaged quantities a^α, $\mathbf{b}^\alpha$

and c^α we have

$$\frac{\partial}{\partial t}(m^\alpha a^\alpha) + \nabla\cdot(m^\alpha \mathbf{b}^\alpha) + m^\alpha c^\alpha + d^\alpha = 0 \tag{2.5}$$

where d^α is implicitly dependent on a, and $\mathbf{b}$ is determined from the relation

$$\iint\limits_{\Gamma\{\alpha\}} (\mathbf{n}\cdot\mathbf{b} - w_n a)ds = \iiint\limits_{\Omega} d^\alpha d\upsilon$$

Thus, we have the following result.

Lemma. *Let some conservation law of the form* (2.1) *be specified. Then, the average values* a^α, $\mathbf{b}^\alpha$, c^α ($\alpha = 0, 1, \ldots, n$) *satisfy the divergence-type inhomogeneous* Eq. (2.5).

The differential Eq. (2.5) contains the average value $\mathbf{b}^\alpha$ of the flux $\mathbf{b}$, which is represented in the general case in the form of the sum of the products of several physical quantities. This raises the problem of expressing the average value of a product in terms of the average values of its cofactors.[5] In constructing the model of a porous medium, this problem is solved in the following manner. Each physical quantity f is represented in the form of a sum of the average value f^α and the fluctuation Δf^α (Ref. 1). Now, in the case of the product of two quantities, $\rho\mathbf{v}$, we will show how the following expressions can be obtained from integral relation (1.5) with one additional assumption

$$(\rho\mathbf{v})^\alpha = \rho^\alpha\mathbf{v}^\alpha + \mathbf{\Pi}^\alpha, \quad \mathbf{\Pi}^\alpha = (\Delta\rho\Delta\mathbf{v})^\alpha$$

In fact, the following chain of equalities holds

$$\iiint\limits_{\Omega\{\alpha\}} \rho d\upsilon = \iiint\limits_{\Omega} m^\alpha(\rho\mathbf{v})^\alpha d\upsilon = \iiint\limits_{\Omega} m^\alpha(\rho^\alpha\mathbf{v}^\alpha + (\rho^\alpha\Delta\mathbf{v})^\alpha + (\Delta\rho\mathbf{v}^\alpha)^\alpha + (\Delta\rho\Delta\mathbf{v})^\alpha)d\upsilon =$$

$$= \iiint\limits_{\Omega} m^\alpha(\rho^\alpha\mathbf{v}^\alpha + (\Delta\rho\Delta\mathbf{v})^\alpha)d\upsilon + J_\rho + J_\upsilon, \quad J_\rho = \iiint\limits_{\Omega\{\alpha\}} (\Delta\rho)\mathbf{v}^\alpha d\upsilon, \quad j_\upsilon = \iiint\limits_{\Omega\{\alpha\}} \rho^\alpha\Delta\mathbf{v}^\alpha d\upsilon$$

$$J_\upsilon = \iiint\limits_{\Omega\{\alpha\}} \rho^\alpha(\Delta\mathbf{v})^\alpha d\upsilon$$

Since the quantities $\Delta\rho$ and $\Delta\mathbf{v}$ oscillate rapidly about the zero value, it may be postulated, by analogy with the Riemann–Lebesgue lemma from the theory of Fourier series,[7] that the integrals J_ρ and J_υ will also be close to zero and may be neglected. In this case we obtain the equality

$$\iiint\limits_{\Omega} m^\alpha(\rho^\alpha\mathbf{v}^\alpha + (\Delta\rho\Delta\mathbf{v})^\alpha)d\upsilon = 0$$

By virtue of the arbitrariness of Ω and the condition $m^\alpha > 0$, the representation of $(\rho\mathbf{v})^\alpha$ in terms of $\rho^\alpha\mathbf{v}^\alpha$ and $\mathbf{\Pi}^\alpha$ given above follows from this relation. Quantities like $\mathbf{\Pi}^\alpha$ are generally called fluctuation amplitudes. Following similar reasoning, we can represent the average value of the product of any finite number of quantities in the form of the sum of two terms: the product of the average values of these quantities and the fluctuation amplitudes Π_ρ, Π_υ, Π_e and Π_s as functions of the velocities, thermodynamic quantities and possibly their gradients.

Substituting the corresponding physical quantities into Eq. (2.5) in place of α^α, $\mathbf{b}^\alpha$ and c^α, we obtain differential equations for the averaged quantities. The superscript α is henceforth omitted wherever its omission does not lead to misunderstandings, and the relations are written in the same form for all phases. We have

$$\begin{aligned}
&\frac{\partial}{\partial t}(m\rho) + \nabla \cdot (m\rho\mathbf{v}) = \Pi_\rho \\
&\frac{\partial}{\partial t}(m\rho\mathbf{v}) + \nabla \cdot (m\rho\mathbf{v} \otimes \mathbf{v}) = \nabla \cdot (m\boldsymbol{\sigma}) + m\rho\mathbf{f} + \mathbf{F} + \boldsymbol{\Pi}_v \\
&\frac{\partial}{\partial t}(m\rho(e + \mathbf{v}^2/2)) + \nabla \cdot (m\rho(e + \mathbf{v}^2/2)\mathbf{v}) = \nabla \cdot (m\boldsymbol{\sigma}\mathbf{v}) - \nabla \cdot (m\mathbf{q}) + m\rho\mathbf{f} \cdot \mathbf{v} + W + Q + \Pi_e \\
&\frac{\partial}{\partial t}(m\rho s) + \nabla \cdot (m\rho s\mathbf{v}) = -\nabla \cdot \frac{m\mathbf{q}}{T} - \frac{m\mathbf{q}}{T}\frac{\nabla T}{T} + \frac{Q + \tilde{q}}{T} + \Pi_s
\end{aligned} \tag{2.6}$$

Here $e + \mathbf{v}^2/2$ is the total energy, $\tilde{q}$ corresponds to the uncompensated heat, $\tilde{q} \geq 0$, and in irreversible processes $\tilde{q} > 0$ (Ref. 8). The first equality in (2.6) corresponds to the law of conservation of mass, the second corresponds to the law of conservation of momentum, the third corresponds to law of the conservation of energy, and the fourth corresponds to the second law of thermodynamics.

The quantities $\mathbf{F}$, W and Q correspond to d^α in Eq. (2.5) and are introduced on the basis of representability condition (1.6) of follows:

$$\begin{aligned}
&\iint\limits_{\Gamma\{\alpha\}} \boldsymbol{\sigma} \cdot \mathbf{n} ds = \iiint\limits_{\Omega} \mathbf{F}^\alpha dv, \quad \iint\limits_{\Gamma\{\alpha\}} (\boldsymbol{\sigma} \cdot \mathbf{v}) \cdot \mathbf{n} ds = \iiint\limits_{\Omega} W^\alpha dv \\
&\iint\limits_{\Gamma\{\alpha\}} \mathbf{q} \cdot \mathbf{n} ds = \iiint\limits_{\Omega} Q^\alpha dv, \quad \iint\limits_{\Gamma\{\alpha\}} \frac{\mathbf{q} \cdot \mathbf{n}}{T} ds = \iiint\limits_{\Omega} \frac{Q^\alpha}{T^\alpha} dv
\end{aligned} \tag{2.7}$$

We will describe how the fourth relation in (2.7) was obtained in greater detail. When the representability condition is taken into account, the expression on the left-hand side of this relation can only be obtained in the form

$$\iint\limits_{\Gamma\{\alpha\}} \left(\frac{\mathbf{q}}{T}\right) \cdot \mathbf{n} ds = \iiint\limits_{\Omega} \hat{Q}^\alpha dv$$

We now assume that the temperature is the same over the entire body. Then, the left-hand side of the last relation can be transformed as follows

$$\iint\limits_{\Gamma\{\alpha\}} \left(\frac{\mathbf{q}}{T}\right) \cdot \mathbf{n} ds = \frac{1}{T} \iint\limits_{\Gamma\{\alpha\}} (\mathbf{q} \cdot \mathbf{n}) ds = \frac{1}{T} \iiint\limits_{\Omega} Q^\alpha dv = \iiint\limits_{\Omega} \frac{Q^\alpha}{T} dv = \iiint\limits_{\Omega} \frac{Q^\alpha}{T^\alpha} dv$$

To satisfy this relation, $\hat{Q}^\alpha$ must be taken in the form Q^α/T^α.

The quantity $\mathbf{F} = (\mathbf{F}^0, \mathbf{F}^1, \ldots \mathbf{F}^n)$ is called the phase-interaction force, $\boldsymbol{W} = (W^0, W^1, \ldots, W^n)$ is the work of the phase-interaction forces, and $\boldsymbol{Q} = (Q^0, Q^1, \ldots, Q^n)$ characterizes the intensity of the interphase heat transfer. To determine these quantities, some physical assumptions must be made.

We will present some estimates of the fluctuation amplitudes. In a porous medium, as the estimates in Ref. 1 show, Π_v is fairly small and is usually neglected. We shall henceforth make this assumption. In overwhelming majority of applications, motions of porous media under which the density fluctuations Π_ρ are small, i.e., $|\Delta\rho/\rho| \ll 1$, are considered. In this case, they can be neglected, and the mass conservation equation can be written in a form that is identical to the first equality in (2.6) with zero right-hand side. In addition, the average velocity can then be considered equal to the weighted-average velocity with a high degree of accuracy:

$$\mathbf{v}_{sr} = \int\limits_{\delta v\{\alpha\}} \rho\mathbf{v} dv / \int\limits_{\delta v\{\alpha\}} \rho dv$$

In the general case, the fluctuation terms Π_s in the last equality in (2.6) cannot be assumed to be small. In fact, according to the definition of Π_s, we have

$$\Pi_s^\alpha = m^\alpha\left(\frac{\mathbf{q}\cdot\nabla T}{T^2}\right)^\alpha - \frac{m^\alpha\mathbf{q}^\alpha\cdot\nabla T^\alpha}{(T^2)^\alpha} = \Pi_1 + \frac{m^\alpha\mathbf{q}^\alpha}{(T^2)^\alpha}\cdot((\nabla T)^\alpha - \nabla T^\alpha)$$

Here Π_1 depends on the fluctuation of the heat flux, the temperature and the temperature gradient. Since the temperature gradients in a porous medium can be fairly large, the right-hand side of the last equality can be so great that it cannot be neglected *a priori*.

We will estimate the fluctuations of the internal energy. It is generally known that the internal energy of solid and fluid bodies depends linearly on the temperature over a fairly wide range. In this case, the energy fluctuations Π_e will be of the order of $|\nabla T|$, and thus they too cannot be neglected *a priori*. At the same time, they are considerably smaller than the entropy fluctuations because $\Pi_e \sim \Pi_s/|\nabla T|$. Therefore, it may be assumed that $\Pi_e + \Pi_s \approx \Pi_s$. This relation is used in Section 5 to determine Π_s.

Remark. System (2.6) must be supplemented by relations that follow from the angular momentum equation. We will assume that there are no polarity effects and that the tensor $\boldsymbol{\sigma}$ is consequently symmetric in the original porous medium. It would be incorrect to conclude from this that the average stress tensor $\boldsymbol{\sigma}^0$ obtained after averaging will also be symmetric. In fact, we take the relation which follows from the condition that the torque produced by all of the external forces vanishes in a state of equilibrium

$$\nabla_i(\sigma^{ij}\varepsilon_j \times \mathbf{r}) + \rho\mathbf{f}\times\mathbf{r} = 0$$

Here the $\boldsymbol{\varepsilon}_j$ ($j = 1, 2, 3$) are the basis vectors, and $\mathbf{r} = \mathbf{r}(x)$ is the radius vector.

According to the lemma, this equation leads to the following equation in the averaged quantities

$$(\nabla\cdot(m^0\boldsymbol{\sigma}^0) + m^0\rho^0\mathbf{f}^0)\times\mathbf{r} + m^0\boldsymbol{\omega} + \mathbf{d}_M = \boldsymbol{\Pi}_M \tag{2.8}$$

Here $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is a vector with the components $\omega_i = e_{ijk}((\sigma^0)^{kj} - (\sigma^0)^{jk})$, e_{ijk} denotes the components of the antisymmetric Levi-Civita tensor, $\boldsymbol{\Pi}_M$ denotes the fluctuations, which are equal to zero in the equilibrium state, and $\mathbf{d}_M$ is determined from the relation

$$\iint\limits_{\Gamma\{0\}} (\sigma\cdot\mathbf{n})\times\mathbf{r}\,ds = \iiint\limits_{\Omega}\mathbf{d}_M\,dv$$

and has the meaning of the density of the torque produced by the forces of interaction between the solid and fluid phases.

In the equilibrium state the first term on the left-hand side of Eq. (2.8) is identically equal to zero. Then, from Eq. (2.8) we obtain

$$m\boldsymbol{\omega} + \mathbf{d}_M = 0$$

If $\mathbf{d}_M \equiv 0$, we also have $\omega \equiv 0$. Otherwise, polarity effects will appear in the model as a result of the averaging. The corresponding experimental data would be needed to provide a quantitative description of these effects. However, for the porous materials that should be encountered in geomechanics, it may be assumed with a high degree of confidence that $\mathbf{d}_M = 0$. For this reason, we shall henceforth neglect the polarity effects and assume that $\boldsymbol{\sigma}$ is a symmetric tensor.

Taking into account the equations of continuity, we can express the left-hand sides of equalities (2.6) in terms of total derivatives, for example,

$$m\rho\frac{d\mathbf{v}}{dt} = \frac{\partial(m\rho\mathbf{v})}{\partial t} + \nabla\cdot(m\rho\mathbf{v}\otimes\mathbf{v})$$

Instead of the third equality in (2.6), we can take a linear combination of the second and third equalities in this system of equations with the coefficients −**v**, 1. We then obtain the relation

$$m\rho\frac{de}{dt} = m\boldsymbol{\sigma} : \mathbf{e}(\mathbf{v}) - \nabla\cdot(m\mathbf{q}) + Q + W - \mathbf{F}\cdot\mathbf{v};\quad \frac{de}{dt} = \frac{\partial e}{\partial t} + \mathbf{v}\cdot\nabla e \tag{2.9}$$

which contains a total derivative with respect to time that depends on the velocity of the phase.

We will now obtain a relation that will play a key role below. For this purpose, we take a linear combination of Eq. (2.9) and last equality in (2.6) with coefficients 1, $-T$. We then have

$$m\rho\left(\frac{de}{dt} - T\frac{ds}{dt}\right) = m\boldsymbol{\sigma} : \mathbf{e}(\mathbf{v}) + W - \mathbf{F}\cdot\mathbf{v} - \tilde{q} - \Pi_s T \tag{2.10}$$

which, like Eq. (2.9), holds for any phase and any motions of the porous medium.

Along with the laws of conservation of mass, momentum and energy, the kinematic relation[8]

$$\frac{d\boldsymbol{\varepsilon}}{dt} = \mathbf{e}(\mathbf{v}) - \boldsymbol{\varepsilon}\cdot(\nabla\mathbf{v})^t - \nabla\mathbf{v}\cdot\boldsymbol{\varepsilon} \tag{2.11}$$

which cannot be represented in the divergence form, should hold in the solid phase. Section 7 presents a linearized relation of the type (2.11), which has the divergence form.

To conclude this section, we will introduce some additional notation. Relations that contain summations of average values of physical quantities over all the phases of a porous medium, for example, $\sum_\alpha m^\alpha e^\alpha$, $\sum_\alpha m^\alpha\rho^\alpha e^\alpha$, are obtained below.

To simplify the way in which these expressions are written, let us consider some sequence p of $n+1$ values, i.e., $p=(p^0, p^1, \ldots, p^n)$, in the general case and use $\langle p\rangle$ and $\langle p\rangle_+$ to denote the following sums

$$\langle p\rangle = \sum_{\alpha=0}^{n} p^\alpha,\quad \langle p\rangle_+ = \sum_{\alpha=1}^{n} p^\alpha \tag{2.12}$$

The subscripted plus sign will be omitted wherever it is clear that the sum is taken over all positive numbers. Next, let the two sequences $p=(p^0, p^1, \ldots, p^n)$ and $q=(q^0, q^1, \ldots, q^n)$ be given. We will use pq to denote their product

$$pq = (p^0q^0, p^1q^1, \ldots, p^nq^n) \tag{2.13}$$

The product of any number of sequences is defined similarly. For example,

$$m\rho e = (m^0\rho^0e^0, m^1\rho^1e^1, \ldots, m^n\rho^ne^n)$$

We will use ∇p to denote the sequence

$$\nabla p = (\nabla p^0, \nabla p^1, \ldots, \nabla p^n)$$

Then, for example,

$$\langle\nabla\cdot(m\rho e\mathbf{v})\rangle = \sum_{\alpha=0}^{n}\nabla\cdot(m^\alpha\rho^\alpha e^\alpha\mathbf{v}^\alpha)$$

Now let $\alpha\in N=(0, 1, \ldots, n)$. We will use $[\alpha]$ and (α) to denote the sets of all integers from zero and unity, respectively, to n, excluding α:

$$[\alpha] = (\beta|\beta\neq\alpha, \beta\in[0, 1, \ldots, n]),\quad (\alpha) = (\beta|\beta\neq\alpha, \beta\in[1, 2, \ldots, n])$$

We present an identity that is used below. Let the three sequences $(a^1, a^2, \ldots, a^n)$, $(b^1, b^2, \ldots, b^n)$, $(c^1, c^2, \ldots, c^n)$ be given. Then the equality

$$\langle c\rangle\langle cba\rangle = \langle cb\rangle\langle ca\rangle + \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} c^\alpha c^\beta (b^\alpha - b^\beta)(a^\alpha - a^\beta) \tag{2.14}$$

holds. According to (2.12) and (2.13),

$$\langle c\rangle = \sum_{\alpha=1}^{n} c^\alpha, \quad \langle cba\rangle = \sum_{\alpha=1}^{n} c^\alpha b^\alpha a^\alpha, \quad \langle cb\rangle = \sum_{\alpha=1}^{n} c^\alpha b^\alpha, \quad \langle ca\rangle = \sum_{\alpha=1}^{n} c^\alpha a^\alpha$$

To prove the validity of equality (2.14), we open the parentheses in the double sum and combine similar terms. Let us examine one special case, when the following conditions are satisfied

$$\langle b\rangle = 0, \quad c^\alpha = 1, \quad \alpha = 1, 2, \ldots, n \tag{2.15}$$

Then, from (2.14) we obtain

$$\langle ba\rangle = \frac{1}{2n}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} (b^\alpha - b^\beta)(a^\alpha - a^\beta) \tag{2.16}$$

3. The relation between averaged quantities

We will now obtain two integral equations that relate the limit values of the physical quantities on the phase boundary $\Gamma(\alpha)$ to the average values. The limit values of these quantities on the boundary $\partial\Omega\{\alpha\}$ of the region $\Omega\{\alpha\}$ are denoted by the subscript α, for example,

$$\mathbf{v}_\alpha = \lim_{s\to 0,\, s<0} \mathbf{v}(x + \mathbf{n}s, t); \quad x \in \partial\Omega\{\alpha\}$$

For this purpose, we take the region $\Omega\{\alpha\}$ and calculate the surface integral of $p_\alpha \mathbf{n}_\alpha$ over $\partial\Omega\{\alpha\}$

$$\iint_{\partial\Omega\{\alpha\}} p_\alpha \mathbf{n}_\alpha dS = \iiint_{\Omega\{\alpha\}} \nabla p dv = \iiint_{\Omega} m^\alpha \nabla p^\alpha dv \tag{3.1}$$

The first equality follows from the Gauss formula, and the second equality follows from the definition of averaged quantities. On the other hand, we transform the left-hand side of this equation by taking into account definition (1.7) of averaged quantities over a surface. We obtain

$$\begin{aligned}\iint_{\partial\Omega\{\alpha\}} p_\alpha \mathbf{n}_\alpha ds &= \iint_{\Gamma\{\alpha\Omega\}} p_\alpha \mathbf{n}_\alpha ds + \iint_{\Gamma\{\alpha\}} p_\alpha \mathbf{n}_\alpha ds = \iint_{\partial\Omega} m^\alpha p^\alpha \mathbf{n} ds + \iint_{\Gamma\{\alpha\}} p_\alpha \mathbf{n}_\alpha ds = \\ &= \iiint_{\Omega} \nabla(m^\alpha p^\alpha) dv + \iint_{\Gamma\{\alpha\}} p_\alpha \mathbf{n}_\alpha ds\end{aligned} \tag{3.2}$$

Equating the right-hand sides of (3.1) and (3.2), calculating $\nabla(m^\alpha p^\alpha)$ using the Leibnitz formula, and then combining terms, we obtain the equation

$$\iint_{\Gamma\{\alpha\}} p_\alpha \mathbf{n}_\alpha ds = -\iiint_{\Omega} \nabla m^\alpha p^\alpha dv + \iiint_{\Omega} m^\alpha (\nabla p^\alpha - \nabla(p^\alpha)) dv \tag{3.3}$$

from which it follows that the equality

$$-\iint\limits_{\Gamma\{\alpha\}} p_\alpha \mathbf{n}_\alpha ds = \iiint\limits_{\Omega} \nabla m^\alpha p^\alpha dv \tag{3.4}$$

holds if and only if the relation

$$\iiint\limits_{\Omega} m^\alpha(\nabla p^\alpha - \nabla(p^\alpha))dv = 0 \tag{3.5}$$

holds, i.e., the differentiation and the taking of the average value are commutative in the sense indicated above.

Exactly the same result can also be obtained for differentiation with respect to time after using the relation

$$\iint\limits_{\Gamma\{\alpha\}} p_\alpha(\mathbf{v}_\alpha \cdot \mathbf{n}_\alpha)ds = \iiint\limits_{\Omega} p^\alpha \frac{\partial m^\alpha}{\partial t} dv + \iiint\limits_{\Omega} m^\alpha\left(\frac{\partial p^\alpha}{\partial t} - \left(\frac{\partial p}{\partial t}\right)^\alpha\right)dv \tag{3.6}$$

which follows from the integral definition of averaged quantities and the formula for differentiating the integral over the variable region with respect to a parameter.

It follows from (3.6) that the equality

$$\iint\limits_{\Gamma\{\alpha\}} p_\alpha(\mathbf{v}_\alpha \cdot \mathbf{n}_\alpha)ds = \iiint\limits_{\Omega} p^\alpha \frac{\partial m^\alpha}{\partial t} dv \tag{3.7}$$

holds if and only if the relation

$$\iiint\limits_{\Omega} m^\alpha\left(\frac{\partial p^\alpha}{\partial t} - \left(\frac{\partial p}{\partial t}\right)^\alpha\right)dv = 0 \tag{3.8}$$

holds, i.e., the differentiation with respect to time and the taking of the average are commutative in the integral sense.

Thus, we arrive at the following conclusion. The surface integrals over the boundary $\Gamma\{\alpha\}$ can be expressed in terms of the volume integrals of the averaged quantities over the entire region Ω using formulae (3.4) and (3.7) if relations (3.5) and (3.8) hold. We shall henceforth assume that relations (3.5) and (3.8) are valid. On the other hand, relations (3.3) and (3.6) open the way for devising a more general theory, if we adopt other, more exact assumptions instead of (3.5) and (3.8).

From identity (2.16) and formulae (3.7) we have

$$\begin{aligned}\frac{1}{2}\sum_{\alpha=0}^{n}\sum_{\beta\in[\alpha]}\iint\limits_{\Gamma\{\alpha\beta\}}(p_\alpha - p_\beta)\mathbf{v}_\alpha \cdot \mathbf{n}_\alpha ds &= \sum_{\alpha=0}^{n}\iint\limits_{\Gamma\{\alpha\}} p_\alpha \mathbf{v}_\alpha \cdot \mathbf{n}_\alpha ds = \sum_{\alpha=0}^{n}\iiint\limits_{\Omega} p^\alpha \frac{\partial m^\alpha}{\partial t} dv = \\ &= \iiint\limits_{\Omega} m\left\langle \frac{\partial \theta}{\partial t} p\right\rangle_+ dv = \frac{1}{2n}\iiint\limits_{\Omega} m \sum_{\alpha,\beta=1}^{n}(p^\alpha - p^\beta)\left(\frac{\partial\theta_\alpha}{\partial t} - \frac{\partial\theta_\beta}{\partial t}\right)dv\end{aligned} \tag{3.9}$$

Another important relation follows from identity (2.16) and formula (3.4)

$$\begin{aligned}\frac{1}{2}\sum_{\alpha=0}^{n}\sum_{\beta\in[\alpha]}\iint\limits_{\Gamma\{\alpha\beta\}}(p_\alpha - p_\beta)\mathbf{n}_\alpha ds &= \sum_{\alpha=0}^{n}\iint\limits_{\Gamma\{\alpha\}} p_\alpha \mathbf{n}_\alpha ds = -\sum_{\alpha=0}^{n}\iiint\limits_{\Omega} p^\alpha \nabla m^\alpha dv = \\ &= -\iiint\limits_{\Omega} m\langle p\nabla\theta\rangle_+ dv = -\frac{1}{2n}\iiint\limits_{\Omega} m \sum_{\alpha,\beta=1}^{n}(p^\alpha - p^\beta)(\nabla\theta_\alpha - \nabla\theta_\beta)dv\end{aligned} \tag{3.10}$$

In (3.9) and (3.10) the last equalities between the volume integrals follow from identity (2.16), which is a special case of identity (2.14) when condition (2.15) holds.

Integral relation (3.10) yields the relation between pressure jumps on the boundaries of the media and the differences between the phase pressures $p^\alpha - p^\beta$ in the fluids. These relations can be used to take surface phenomena into account. In fact, if we postulate that the phase pressures cannot be arbitrary and must satisfy some conditions, it follows from (3.10) that the pressure jumps on the boundaries also cannot be arbitrary. Conversely, since the pressure jumps on the boundaries should satisfy Laplace's law, the differences between the phase pressures cannot be arbitrary. On the basis of these arguments, it is natural to assume, according to formula (3.10), that in a state of thermodynamic equilibrium the differences between the phase pressures should satisfy the conditions

$$p^\alpha - p^\beta = p_c^{\alpha\beta}(\theta_1, \dots, \theta_{n-1}) \tag{3.11}$$

where the $p_c^{\alpha\beta}$ are some functions, which must be determined from of experimental data. In Section 6 we will return to (3.11) to describe the surface phenomena.

If we compare equalities (3.4) and (3.7) to the representability condition (1.6), we can arrive at the following condition. Relations (3.4) and (3.7) not only agree with condition (1.6), but also yield an explicit expression for the integrand in the volume integrals on the right-hand sides of (3.4) and (3.7). Equalities (3.4) and (3.7) can also be established from physical arguments and the dimension theory. This approach is presented in Section 4.

4. Expansions of $\mathbf{F}^\alpha$, W^α and Q^α

Let us take the first step toward determining $\mathbf{F}^\alpha$, W^α and Q^α. We shall present arguments that provide a basis for assuming that they can be represented in the form

$$\mathbf{F}^\alpha = -p^\alpha \nabla m^\alpha + \mathbf{R}^\alpha, \quad \mathbf{R}^\alpha = \sum_{\beta \in [\alpha]} \mathbf{R}^{\alpha\beta}, \quad \mathbf{R}^{\alpha\beta} = -\mathbf{R}^{\beta\alpha}, \quad \alpha \neq \beta \tag{4.1}$$

$$W^\alpha = p^\alpha \frac{\partial m^\alpha}{\partial t} + D^\alpha, \; D^\alpha = \sum_{\beta \in [\alpha]} D^{\alpha\beta} = \sum_{\beta \in [\alpha]} \mathbf{R}^{\alpha\beta} \cdot \mathbf{v}^\beta, \; \alpha = 1, 2, \dots, n, \; D^0 = \mathbf{R}^0 \cdot \mathbf{v}^0 \quad (4.2) \tag{4.2}$$

$$Q^\alpha = \sum_{\beta \in [\alpha]} Q^{\alpha\beta}, \quad Q^{\alpha\beta} = -Q^{\beta\alpha}, \quad \alpha = 0, 1, \dots, n \tag{4.3}$$

In fact, the stress tensor in each phase can be represented as a sum of the spherical part p and the deviator $\boldsymbol{\tau}$ for the solid phase (or the viscous stress tensor for the fluid phases). Then the integral over the boundary $\Gamma\{\alpha\}$ between the α phase and the adjacent phases can be represented in the form of the sum

$$\iint_{\Gamma\{\alpha\}} \boldsymbol{\sigma}_n ds = \iint_{\Gamma\{\alpha\}} p\mathbf{n} ds + \iint_{\Gamma\{\alpha\}} \boldsymbol{\tau}_n ds, \quad \boldsymbol{\sigma}_n = \boldsymbol{\sigma} \cdot \mathbf{n}, \quad \boldsymbol{\tau}_n = \boldsymbol{\tau} \cdot \mathbf{n} \tag{4.4}$$

We transform the first term on the right-hand side of equality (4.4) into a volume integral over the region Ω according to (3.4), and we write the second term on the basis of the representability condition (1.6) in the form of the volume integral

$$\iint_{\Gamma\{\alpha\}} \boldsymbol{\tau}_n ds = \iiint_{\Omega} \mathbf{R}^\alpha dv \tag{4.5}$$

(the quantity $\mathbf{R}^\alpha$ has the meaning of the total force that the adjacent phases apply to the α phase at the boundary $\Gamma\{\alpha\}$). We obtain

$$\iint_{\Gamma\{\alpha\}} \boldsymbol{\sigma}_n ds = -\iiint_{\Omega} p^\alpha \nabla m^\alpha dv + \iiint_{\Omega} \mathbf{R}^\alpha dv \tag{4.6}$$

Next, based on physical arguments, we may conclude that the surface integrals over the boundary $\Gamma\{\alpha\beta\}$, like those over the boundary $\Gamma\{\alpha\}$, can be transformed into the volume integrals

$$\iint_{\Gamma\{\alpha\beta\}} \boldsymbol{\tau}_n ds = \iiint_{\Omega} \mathbf{R}^{\alpha\beta} dv$$

Taking this relation into account, as well as the relation

$$\Gamma\{\alpha\} = \bigcup_{\beta} \Gamma\{\alpha\beta\}, \quad \beta \in [\alpha]$$

we reduce the left-hand side of equality (4.5) to the form

$$\iint_{\Gamma\{\alpha\}} \boldsymbol{\tau}_n ds = \sum_{\beta\in[\alpha]} \iint_{\Gamma\{\alpha\beta\}} \boldsymbol{\tau}_n ds = \iiint_{\Omega} \left(\sum_{\beta\in[\alpha]} \mathbf{R}^{\alpha\beta} \right) dv \tag{4.7}$$

From (4.5)–(4.7) we have the second equality in (4.1).

The $\mathbf{R}^{\alpha\beta}$ can be regarded as the forces that the β phase applies to the α phase at the boundary $\Gamma\{\alpha\beta\}$. These quantities are governed by the viscous friction forces in the fluid phases. According to Newton's third law, they must satisfy the third relation in (4.1). No surface tension appears here.

In order to arrive at relation (4.2), we start out from the equality

$$\iint_{\Gamma\{\alpha\}} \boldsymbol{\sigma}_n \mathbf{v} ds = \iint_{\Gamma\{\alpha\}} p \mathbf{v}_n ds + \iint_{\Gamma\{\alpha\}} \boldsymbol{\tau}_n \cdot \mathbf{v} ds$$

We transform the first integral on the right-hand side into a volume integral over the region Ω according to (3.7), and we transform the second integral into a volume integral based on the representability condition

$$\iint_{\Gamma\{\alpha\}} \boldsymbol{\tau}_n \cdot \mathbf{v} ds = \iiint_{\Omega} D^{\alpha} dv$$

Here D^{α} is a quantity that may be regarded as the rate of the total work performed on the α phase by the adjacent phases at the boundary $\Gamma\{\alpha\}$. Therefore, using the same arguments as above, the quantity D^{α} for fluid phases may be represented in the form of the sum

$$D^{\alpha} = \sum_{\beta\in[\alpha]} D^{\alpha\beta}, \quad \iint_{\Gamma\{\alpha\beta\}} \boldsymbol{\tau}_n \cdot \mathbf{v} ds = \iiint_{\Omega} D^{\alpha\beta} dv \tag{4.8}$$

The $D^{\alpha\beta}$, where $\alpha \neq \beta$, have the meaning of the work performed on the α phase by the β phase at the boundary $\Gamma\{\alpha\beta\}$. From physical arguments it is clear that the $D^{\alpha\beta}$ should depend both on the forces $\mathbf{R}^{\alpha\beta}$ and on $\mathbf{v}^{\alpha}$. In addition, from the dimension theory it follows that the relation $D^{\alpha\beta} \sim \mathbf{R}^{\alpha\beta} \cdot \mathbf{v}^{\beta}$ should hold.

We note one more property that the integral on the left-hand side of the second equality in (4.8) should have. We will assume that the velocities of all the adjacent phases are constant in each of the regions $\Omega\{\beta\}$ bordering $\Omega\{\alpha\}$ and that the velocities are continuous on the boundaries of the fluid phases. We then obtain

$$\iint_{\Gamma\{\alpha\}} \boldsymbol{\tau}_n \cdot \mathbf{v} ds = \sum_{\beta\in[\alpha]} \int \int_{\Gamma\{\alpha\beta\}} \boldsymbol{\tau}_n \cdot \mathbf{v} ds = \sum_{\beta\in[\alpha]} \mathbf{v}^{\beta} \cdot \iint_{\Gamma\{\alpha\beta\}} \boldsymbol{\tau}_n ds =$$

$$= \sum_{\beta\in[\alpha]} \mathbf{v}^{\beta} \cdot \iiint_{\Omega} \mathbf{R}^{\alpha\beta} dv = \sum_{\beta\in[\alpha]} \iiint_{\Omega} \mathbf{R}^{\alpha\beta} \cdot \mathbf{v}^{\beta} dv$$

The property just described, as well as relations (4.8) and the physical arguments presented above, can be satisfied, if we assume that the following condition holds

$$\iint\limits_{\Gamma\{\alpha\}} \boldsymbol{\tau}_n \cdot \mathbf{v}ds = \iiint\limits_{\Omega} \mathbf{R}^{\alpha\beta} \cdot \mathbf{v}^{\beta} dv \tag{4.9}$$

Then, the second relation in (4.2) follows from this condition.

We will now calculate the total work of the phase-interaction forces due to all the fluid phases and isolate the part that is stipulated by the interaction of the solid phase with all the fluid phases from this sum. We will also assume that the dissipation of mechanical energy on the boundary $\Gamma\{0\}$ between the solid phase and the adjacent fluid phases is small. Then this part of the total work should be equal to D^0 apart from its sign. We now perform the transformations indicated and, as a result, we obtain the representation for D^0

$$\sum_{\alpha=1}^{n} D^{\alpha} = \sum_{\alpha=1}^{n} \sum_{\beta=0}^{n} \mathbf{R}^{\alpha\beta} \cdot \mathbf{v}^{\beta} = \sum_{\alpha=1}^{n} \mathbf{R}^{\alpha 0} \cdot \mathbf{v}_0 + \sum_{\alpha,\beta=1}^{n} \mathbf{R}^{\alpha\beta} \cdot \mathbf{v}^{\beta} = -D^0 + \sum_{\alpha,\beta=1}^{n} \mathbf{R}^{\alpha\beta} \cdot \mathbf{v}^{\beta}$$

$$D^0 = -\sum_{\alpha=1}^{n} \mathbf{R}^{\alpha 0} \cdot \mathbf{v}^0 = \left(\sum_{\alpha=1}^{n} \mathbf{R}^{\alpha 0}\right) \cdot \mathbf{v}^0 = \mathbf{R}^0 \cdot \mathbf{v}^0$$

In the case of a two-phase medium ($n = 1$), equalities (4.1)–(4.3) lead to relations that are identical, apart from the notation, to the representations for the phase-interaction forces and heat-exchange forces and work first proposed by Rakhmatulin.[9]

Relations (3.4) and (3.7) were obtained under the assumption that the differentiation and the taking of the average in the integral sense are commutative, i.e., equalities (3.5) and (3.8) hold. We can also arrive at relations (3.4) and (3.7) using arguments of the dimension theory. More specifically, from the representability condition we have the relations

$$\iint\limits_{\Gamma\{\alpha\}} p\mathbf{n}ds = \iiint\limits_{\Omega} A^{\alpha} dv, \quad \iint\limits_{\Gamma\{\alpha\}} p v_n ds = \iiint\limits_{\Omega} B^{\alpha} dv \tag{4.10}$$

Here $|A^{\alpha}| \sim p^{\alpha}/L$, L is a quantity that has the meaning of the characteristic dimension, $|B^{\alpha}| \sim p^{\alpha}/M$, and M is a quantity that has the dimension of time.

From physical arguments we can arrive at the conclusion that the integrals on the left-hand sides of relations (4.10) are greater, the greater the number of capillaries in the region Ω. At the same time, the relative volume m^{α} of the α phase is also greater, the greater the number of capillaries in the region Ω. Therefore, it is natural to take ∇m^{α} as $1/L$ and $\partial\alpha/\partial t$ as $1/M$ and to obtain relations (3.4) and (3.7) as a result.

Relations (4.3) are obtained in the following manner. We start out by determining Q^{α} from the third formula in (2.7) and repeat all the arguments that led to the representation of $\mathbf{R}^{\alpha}$ in the form of the sum of the $\mathbf{R}^{\alpha\beta}$.

The $Q^{\alpha\beta}$ can be regarded as the rate of heat flow from the β phase to the α phase on the boundary $\Gamma\{\alpha\beta\}$. Therefore, from the law of the conservation of energy it follows that relations (4.3) should hold.

5. Surface phenomena

When a porous medium moves, the surface-tension forces play an important role. This is explained as follows. The pore space consists of a dense branched network of capillaries and pores. The fluid phases that fill the pore space are in a finely dispersed state. Under these conditions the characteristic dimensions of the finely dispersed fluid particles are much smaller than the characteristic dimensions of macroscopic bodies. Hence it follows that the total surface area of the fluid particles in the pore space is much greater than the surface area of the body. In this case the surface energy may be comparable to the internal energy of the solid and fluid phases, and the surface tension forces will then be comparable with the stresses in the solid and fluid phases.[10] As a result of the action of the surface tension forces, the normal stresses have a discontinuity at the phase boundary. Based on the thermodynamic equilibrium condition, we can obtain Laplace's law,[10,11] which relates the pressure jump $\Delta p = p_r - p_e$ to the mean curvature of the interface (p_r

and p_e are the limits of the hydrostatic pressures on the right-hand and left-hand sides as the spatial coordinates tend to the interface along a normal to the interface). We have

$$\Delta p = \alpha(1/R_1 + 1/R_2) \tag{5.1}$$

Here α is the surface tension, and R_1 and R_2 are the principal radii of curvature of the normal cross sections. They are positive if the normal **n** to the surface coincides with the principal normal of the corresponding normal cross section and are negative if these normals are in opposite directions.

This raises the question of how Laplace's law can be reflected in a model of a porous medium in which there are no phase boundaries. It is not easy to give an answer to this question based on an averaging procedure. First of all, the physical picture of the relative arrangement of the fluid particles is unclear, and we do not know the size distribution of the finely disperse particles or whether it is necessary to take into account the agglomeration and crumbling of the particles. Therefore, we must make some physical assumptions and rely on a phenomenological approach to take the surface phenomena into account. Leverett's law[3] is widely used in the theory of three-phase porous media, in which there are only two fluid phases. This law relates the difference between the phase pressures of the fluid phases to the saturation of one of the fluids:

$$\Delta p = \alpha j(\theta)\cos\psi \tag{5.2}$$

Here ψ is the wetting angle, which is assumed to be known for an assigned pair of fluid phases, and $j(\theta)$ is Leverett's function. An extension of this law to the case in which the number of fluids is equal to three was given in Ref. 4, where it was assumed that the difference between the phase pressures of any two fluids is a function of the saturations:

$$p^\alpha - p^\beta = p_c^{\alpha\beta}(\theta_1, \theta_2), \quad \alpha, \beta = 1, 2, 3, \quad \alpha \neq \beta \tag{5.3}$$

Here we have the trivial relations

$$p_c^{\alpha\beta} = -p_c^{\beta\alpha}, \quad p_c^{12} + p_c^{23} + p_c^{31} = 0 \tag{5.4}$$

We will now move on to derive the relations associated with surface phenomena. We will first introduce the average surface energy e^c and the average surface entropy s^c. For this purpose, we take an arbitrary simply connected region Ω in the porous medium and calculate the surface energy E and the entropy S in it[10]

$$E = \iint\limits_{\Gamma}(\alpha - T\alpha_T)ds, \quad S = -\iint\limits_{\Gamma} T\alpha_{TT}ds \tag{5.5}$$

Here Γ is the set of all surfaces between the phases enclosed in the region Ω, and α_T and α_{TT} are the first and second derivatives of the surface tension with respect to the temperature. Using the representability condition, we express the surface integrals in terms of volume integrals and introduce the quantities e^c and s^c in the following manner

$$\iint\limits_{\Gamma}(\alpha - T\alpha_T)ds = \iiint\limits_{\Omega} m\rho^c e^c dv, \quad -\iint\limits_{\Gamma} T\alpha_{TT}ds = \iiint\limits_{\Omega} m\rho^c s^c dv \tag{5.6}$$

Here the density of the solid phase can be taken as ρ^c.

Now we must find the relation that should be satisfied by e^c and s^c. If we use a phenomenological approach, it must be assumed, in accordance with the first law of thermodynamics, that the increment of the average surface energy de^c is equal to the sum of the elementary work dA and the elementary heat flux dq^c

$$m\rho^c de^c = dA + dq^c \tag{5.7}$$

To resolve the question of the form in which the work dA can be represented, we turn to the following arguments. We calculate the elementary work locally. For this purpose, we take any interface Γ, and we define an elementary area on it with the point x and the area ds. Taking into account the formula for the increment (variation) δs of the area element ds when the area is displaced over the distance $\delta r = v_n\, dt$ and Laplace's law, we obtain an expression for the elementary work of the surface tension forces[10]

$$\delta A = \alpha\delta s = -2\alpha H\delta r ds = -\Delta p v_n ds dt$$

We will calculate the total work performed during the time interval *dt* when the phase boundaries Γ enclosed in the region Ω move. We have

$$\Delta A = \iint_{\Gamma} \delta A = -\iint_{\Gamma} \Delta p v_n ds dt \tag{5.8}$$

It is natural to assume that the surface tension forces on the boundaries between the solid and fluid phases are small and can be neglected. In this case the jumps in the normal stresses on these boundaries will be equal to zero, and the left-hand side of (3.9) will be identical to the right-hand side of (5.8) apart from a multiplier, since

$$(p_\alpha - p_\beta)\mathbf{v}_\alpha \cdot \mathbf{n}_\alpha = -\Delta p v_n$$

Then relation (5.8) can be written in the form

$$\Delta A = \iint_{\Gamma} \delta A = \frac{dt}{2n}\iiint_{\Omega} m \sum_{\alpha, \beta = 1}^{n} (p^\alpha - p^\beta)\left(\frac{\partial \theta_\alpha}{\partial t} - \frac{\partial \theta_\beta}{\partial t}\right) dv$$

from which we have the expression for the elementary work of the surface-tension forces

$$dA = \frac{m}{2n} \sum_{\alpha, \beta = 1}^{n} (p^\alpha - p^\beta)\left(\frac{\partial \theta_\alpha}{\partial t} - \frac{\partial \theta_\beta}{\partial t}\right) dt \tag{5.9}$$

For a three-phase medium ($n = 2$), from (5.9) we obtain

$$dA = \frac{m}{2}(p^1 - p^2)\left(\frac{\partial \theta_1}{\partial t} - \frac{\partial \theta_2}{\partial t}\right) dt = m(p^1 - p^2) d\theta_1, \quad d\theta_1 = \frac{\partial \theta_1}{\partial t} dt$$

which is identical to the formula for the work *dA* previously postulated in Ref. 2.Substituting relation (5.9) into Eq. (5.7), we obtain

$$m\rho^c de^c = \frac{m}{2n} \sum_{\alpha, \beta = 1}^{n} (p^\alpha - p^\beta)(d\theta_\alpha - d\theta_\beta) + dq^c, \quad d\theta_\alpha = \frac{\partial \theta_\alpha}{\partial t} dt \tag{5.10}$$

We connect the increment of the specific surface entropy ds^c to dq^c by the following relation, which is consistent with the second law of thermodynamics

$$m\rho^c T^c ds^c = dq^c + dq' \tag{5.11}$$

Here dq' is a quantity related to the irreversible processes that occur when the interfaces move, and T^c is a quantity that can formally be called the temperature of the surface phase.[2] We next assume that in an equilibrium state the differences between the phase pressures satisfy the law that was previously proposed in Ref. 4 for a three-phase medium and was extended to the case of a multiphase medium in the following form

$$p^\alpha - p^\beta - p_c^{\alpha\beta} = 0, \quad \alpha \neq \beta; \quad \alpha, \beta = 1, 2, \ldots, n \tag{5.12}$$

By analogy with relation (5.3), the following relations between the $p_c^{\alpha\beta}$ should hold

$$p_c^{\alpha\beta} = -p_c^{\beta\alpha}, \quad p_c^{\alpha\beta} = p_c^{\alpha\beta}(\theta_1, \theta_2, \ldots, \theta_{n-1})$$

In addition, the $p_c^{\alpha\beta}$ should satisfy the compatibility conditions that generalize the second equality in (5.4). We shall provide more details about these conditions below.

When a porous medium moves, the boundaries between the fluid phases begin to be displaced, the equilibrium state is violated, and equality (5.12) does not hold. An irreversible process of equalizing the phase pressures begins. During this process, the quantities appearing in equality (5.12) tend to zero. In this case it is natural to associate them with the amount of uncompensated heat dq' and to assume that the following relation holds

$$dq' = \frac{m}{2n}\sum_{\alpha,\beta=1}^{n}(p^{\alpha}-p^{\beta}-p_c^{\alpha\beta})(d\theta_{\alpha}-d\theta_{\beta}) \tag{5.13}$$

Combining equalities (5.13) and (5.11), we obtain the equation for the rate of entropy change

$$m\rho^c T^c ds^c = dq^c + \frac{m}{2n}\sum_{\alpha,\beta=1}^{n}(p^{\alpha}-p^{\beta}-p_c^{\alpha\beta})(d\theta_{\alpha}-d\theta_{\beta}) \tag{5.14}$$

The difference between (5.10) and (5.14) gives the fundamental equation that relates de^c and ds^c

$$\rho^c(de^c - T^c ds^c) = \frac{1}{2n}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}p_c^{\alpha\beta}(d\theta_{\alpha}-d\theta_{\beta}) \tag{5.15}$$

The validity of (5.10) and (5.14) can also be substantiated by mathematical arguments. For this purpose, we turn to formulae (5.5) and (5.6), which were used to introduce e^c and s^c. We calculate the derivative of the total surface energy E of the porous medium enclosed in the region Ω with respect to time

$$\frac{dE}{dt} = \frac{d}{dt}\iint_{\Gamma}(\alpha - T\alpha_T)ds$$

and the derivative of the surface integral, taking into account the formula for differentiating an integral over a variable surface with respect to a parameter (the time).[6] We also assume that $\alpha = \alpha(T)$. We then obtain

$$\frac{d}{dt}\iint_{\Gamma}\hat{\alpha}ds = \iint_{\Gamma}\left(-2\hat{\alpha}Hv_n + \frac{d\hat{\alpha}}{dt}\right)ds$$

where

$$\hat{\alpha} = \alpha - T\alpha_T,\quad H = \frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right),\quad \frac{d\hat{\alpha}}{dt} = \lim_{\tau\to 0}\frac{\hat{\alpha}(t+\tau, x+\mathbf{n}v_n\tau)-\hat{\alpha}(t,x)}{\tau}$$

Taking into account the rule for differentiating a composite function, we have

$$\frac{d\hat{\alpha}}{dt} = c\frac{dT}{dt},\quad c = -T\alpha_{TT} = Ts_T^c$$

The quantity c may be called the surface entropy.

We express the average curvature H in terms of the pressure jumps Δp from Laplace's law and assume that $|T\alpha_T/\alpha| \ll 1$. We obtain

$$\frac{dE}{dt} = -\iint_{\Gamma}\Delta p v_n ds + \iint_{\Gamma}c\frac{dT}{dt}ds \tag{5.16}$$

The first term on the right-hand side of (5.16) is identical to the right-hand side of (5.8), apart from to within the notation. Therefore, within the assumptions made regarding the surface tension forces, this term can be replaced by the right-hand side of (3.9). We transform the second terms on the basis of the representability condition into a volume integral:

$$\iint_{\Gamma}c\frac{dT}{dt}ds = \iiint_{\Omega}\frac{dq^c}{dt}dv \tag{5.17}$$

The quantity dq^c/dt, which is introduced using (5.17), has the meaning of the rate of heat flow to the "surface" phase.

As a result of the transformations just described, relation (5.16) is reduced to the form

$$\frac{dE}{dt} = \frac{1}{2n}\iiint\limits_{\Omega} m \sum_{\alpha,\beta=1}^{n} (p^{\alpha}-p^{\beta})\left(\frac{\partial\theta_{\alpha}}{\partial t}-\frac{\partial\theta_{\beta}}{\partial t}\right)dv + \iiint\limits_{\Omega}\frac{dq^{c}}{dt}dv$$

from which Eq. (5.10) follows.

We can arrive at relation (5.11) in a similar manner if we start out from the definition of the entropy given in the second formula in (5.5). In addition, we assume that the third derivative of α with respect to T is small and can be neglected. Then, after some reduction similar to that performed above, we arrive at the relation

$$\frac{dS}{dt} = \iint\limits_{\Gamma} T\alpha_{TT}Hv_{n}ds - \iint\limits_{\Gamma}\alpha_{TT}\frac{dT}{dt}ds$$

We transform the surface integrals into volume integrals using the representability condition. Then, taking into account the definitions of c and of dq^c/dt based on (5.17), we obtain

$$-\iint\limits_{\Gamma}\alpha_{TT}\frac{dT}{dt}ds = \iiint\limits_{\Omega}\frac{q^{c}}{T^{c}}dv, \quad \iint\limits_{\Gamma}T\alpha_{TT}Hv_{n}ds = \iiint\limits_{\Omega}\frac{q'}{T^{c}}dv \tag{5.18}$$

Relation (5.11) follows from equalities (5.18). However, it needs to be shown that the formal definition of q' (the second formula in (5.18)) does not contradict the second law of thermodynamics, i.e., the condition $q' \geq 0$, should hold. In fact, Hv_n is proportional to Δpv_n, i.e., to the work done in displacing the interfaces per unit time. This work should be sign-definite, since it may be assumed, on the basis of physical arguments, that such motions are irreversible. Then, assuming that α_{TT} does not change sign, we find that the integrand on the left in the second equality in (5.18) is also sign-definite, and q' does not change sign. The inequality $q' \geq 0$ cannot be established rigorously using these arguments.

6. The work of external forces in a porous medium. Calculation of the fluctuations Π_s

We will write the stress tensor in the fluid phases $\boldsymbol{\sigma}^{\alpha}$ in the form of the sum of a spherical tensor, which we shall call the pressure and denote by p^{α}, and the viscous stress tensor $\boldsymbol{\tau}^{\alpha}$. We introduce the average pore pressure $\widehat{p}$ and the effective stress tensor $\boldsymbol{\sigma}$

$$p = \langle\theta p\rangle_{+}, \quad \boldsymbol{\sigma} = m^{0}(\boldsymbol{\sigma}^{0}-\widehat{p})$$

The quantity $\widehat{p}$ has a simple physical meaning. In fact, we shall consider motions of a porous medium under which the viscous stresses are small and can be neglected, i.e., $\tau^{\alpha} \cong 0$ ($\alpha = 1, 2, \ldots, n$). We calculate the force acting on an area element with normal $\mathbf{n}$ and area ds:

$$d\mathbf{f}_{n} = (m^{0}\boldsymbol{\sigma}^{0} + m^{1}p^{1} + \ldots + m^{n}p^{n})\cdot\mathbf{n}ds = (m^{0}\boldsymbol{\sigma}^{0} + m\widehat{p})\cdot\mathbf{n}ds$$

Since $m^0 + m = 1$, it follows from this expression hat the force $d\mathbf{f}_n$ is equal to the average value of two forces that act on the area element chosen: one from the solid phase and one from the fluid phases.

The effective stress tensor has an important property: it vanishes when the stress tensor in the solid phase $\boldsymbol{\sigma}^0$ is spherical and all of its non-zero components are equal to $\widehat{p}$, i.e., the solid phase is in a state of isotropic compression under the action of the external pressure $\widehat{p}$. It is then natural to adopt this state as a reference configuration in the solid phase and to describe the stress state using the effective stress tensor $\boldsymbol{\sigma}$, rather than $\boldsymbol{\sigma}^0$.

Following these arguments, we next examine the motions of a porous medium under which the difference between $\widehat{p}$ and an element of the spherical part p^0 of the stress tensor $\boldsymbol{\sigma}^0$ is small, so that the quantities $\hat{p}$ and p^0 may be considered identical. This hypothesis can be reformulated using the strain tensor (which will be discussed below) and assuming that the strains are small.

Let us find the relation between the fluctuation amplitudes in equality (2.10) and $\mathbf{R}^{\alpha\beta}$. Since the solid phase is assumed to be elastic and since the dissipation of mechanical energy in the fluid phases is governed by the viscous friction forces, the following relations should hold

$$\tilde{q}^0 = 0, \quad \tilde{q}^\alpha = \boldsymbol{\tau}^\alpha : \mathbf{e}(\mathbf{v}^\alpha), \quad \alpha = 1, \ldots, n$$

Next, from (4.1) and (4.2) we can obtain the equality

$$W - \mathbf{F} \cdot \mathbf{v} = p\frac{dm}{dt} + \hat{W}, \quad \hat{W} = (\hat{W}^0, \hat{W}^1, \ldots, \hat{W}^n)$$
$$\hat{W}^0 = 0, \quad \hat{W}^\alpha = \sum_{\beta \in [\alpha]} \mathbf{R}^{\alpha\beta} \cdot (\mathbf{v}^\beta - \mathbf{v}^\alpha), \quad \alpha = 1, \ldots, n$$

The derivative of the relative volumes with respect to time may be represented in the form

$$\frac{dm}{dt} = m\left(p\frac{d}{dt}\left(\frac{1}{\rho}\right) - \nabla \cdot \mathbf{v}\right)$$

Taking the last relations into account, we reduce Eq. (2.10) to the form

$$m\rho\left(\frac{de}{dt} - T\frac{ds}{dt} - p\frac{d}{dt}\left(\frac{1}{\rho}\right)\right) - A = \hat{W} - \Pi_s T$$
$$A = (A^0, A^1, \ldots, A^n), \quad A^0 = \boldsymbol{\sigma} : \mathbf{e}(\mathbf{v}^0), \quad A^\alpha = 0, \quad \alpha = 1, \ldots, n \tag{6.1}$$

We will show that both sides of this equality should vanish. In fact, on the right there is a difference between two quantities, each of which does not depend on the derivatives with respect to time, while the left-hand side of this relation can take either positive or negative values, depending on the sign of the derivatives with respect to time. Taking into account that equality (6.1) must hold for any pressures in the porous medium, we can conclude that both sides of (6.1) must vanish. It follows from the fact that the right-hand side vanishes that

$$\Pi_s^0 = 0, \quad \Pi_s^\alpha = \sum_{\beta \in [\alpha]} \frac{\mathbf{R}^{\alpha\beta}}{T^\alpha} \cdot (\mathbf{v}^\beta - \mathbf{v}^\alpha), \quad \alpha = 1, \ldots, n$$

Summing the left-hand and right-hand sides of (6.1) for all the values of $\alpha = 0, 1, \ldots, n$ and then combining terms, we obtain

$$\left\langle m\rho\frac{de}{dt}\right\rangle = \boldsymbol{\sigma} : \mathbf{e}(\mathbf{v}^0) + \left\langle \frac{m\rho}{\rho_0} p\frac{d\xi}{dt}\right\rangle + \left\langle m\rho T\frac{ds}{dt}\right\rangle \tag{6.2}$$

Here $\xi^\alpha = \rho_0^\alpha/\rho^\alpha - 1$ is a quantity that is called the dilatancy, and ρ_0^α is the value of the density of the α phase in the reference configuration.

Similarly, after summing the equations for the entropy from system (2.6) for all values of $\alpha = 0, 1, \ldots, n$ and adding Eq. (5.14), which takes into account the surface phenomena, to the equality obtained, we obtain

$$\left\langle m\rho\frac{ds}{dt}\right\rangle + m\rho^c\frac{ds^c}{dt} = -\nabla \cdot J + \mathscr{X}_q + \mathscr{X}_R + \mathscr{X}_Q + \mathscr{X}_\vartheta + \mathscr{X}_\tau$$
$$J = \left\langle \frac{m\mathbf{q}}{T}\right\rangle, \quad \mathscr{X}_q = -\left\langle \frac{m\mathbf{q}}{T} \cdot \frac{\nabla T}{T}\right\rangle, \quad \mathscr{X}_R = \left\langle \frac{\hat{W}}{T}\right\rangle, \quad \mathscr{X}_Q = \left\langle \frac{Q}{T}\right\rangle \tag{6.3}$$
$$\mathscr{X}_\theta = \sum_{\alpha, \beta = 1}^{n} Y^{\alpha\beta} y_{\alpha\beta}, \quad \mathscr{X}_\tau = \langle \boldsymbol{\tau} : \mathbf{e}(\mathbf{v})\rangle_+, \quad Y^{\alpha\beta} = \frac{m}{2nT^c}\left(\frac{\partial\theta_\alpha}{\partial t} - \frac{\partial\theta_\beta}{\partial t}\right), \quad y_{\alpha\beta} = p^\alpha - p^\beta - p_c^{\alpha\beta}$$

7. Small strains

All the relations that have been obtained are valid for any strains. Thus, there was no need to indicate explicitly the state in the solid phase that is taken as the initial state (the reference configuration).

The investigation of the motions of a porous medium in the case where the average pore pressure $\langle \theta p \rangle_+$ is large enough for the solid phase to be in a field of strong compressive stresses is the most interesting and important from the point of view of applications. For such motions, it is natural to take the state of isotropic compression, in which the stress tensor is spherical and all its non-zero components are equal to the average pore pressure p_0, as the reference configuration in the solid phase. The effective stress tensor will then be a rank zero tensor in this state.

We will use m_0^α, T_0^α, s_0^α, ρ_0^α, p_0^α to denote the values of the relative volumes, phase temperatures, entropies, densities and pressures in this state. We will now consider the motions of a porous medium under which the strain tensor relative to the reference configuration chosen is small. We will also assume that the convective terms are small. Then the total derivative with respect to time for each phase is equal to the partial derivative with respect to time. We will denote this derivative by putting a dot over the corresponding physical quantity.

We introduce the following quantities

$$\rho = \langle m_0 \rho_0 \rangle, \quad \rho E = \langle m_0 \rho_0 e \rangle$$

$$\eta^\alpha = (m_0^\alpha / m_0)(m_0^0 \xi^0 + m_0 \xi^\alpha), \quad \alpha = 1, 2, \ldots, n; \quad s_\Delta^\alpha = s^\alpha - s_0^\alpha, \quad \alpha = 0, 1, \ldots, n$$

We will now show that the following relations hold when the strains and convective terms are small

$$m^\alpha \cong m_0^\alpha, \quad \rho^\alpha \cong \rho_0^\alpha \tag{7.1}$$

$$\left\langle m\rho \frac{de}{dt} \right\rangle \cong \rho \dot{E}, \quad \frac{d\boldsymbol{\varepsilon}}{dt} \cong \mathbf{e}(\mathbf{v}^0), \quad \left\langle m\rho p \frac{d}{dt}\left(\frac{1}{\rho}\right) \right\rangle \cong \langle p \dot{\eta} \rangle_+ \tag{7.2}$$

The equalities (7.1) are trivial. The first relation in (7.2) follows from the definition of E.

To prove the validity of the second relation in (7.2), we discard the non-linear terms in equality (2.11), average the equation obtained, and apply the lemma. We obtain

$$\frac{\partial (m^0 \boldsymbol{\varepsilon})}{\partial t} - \frac{1}{2}[\nabla (m^0 \mathbf{v}^0) + (\nabla (m^0 \mathbf{v}^0))^t] = \mathbf{d}_\varepsilon \tag{7.3}$$

$$\iiint\limits_{\Omega} \mathbf{d}_\varepsilon dv = \iint\limits_{\Gamma\{0\}} (\mathbf{n} \otimes \mathbf{v}^0 + \mathbf{v}^0 \otimes \mathbf{n} - w_n \boldsymbol{\varepsilon}) ds \tag{7.4}$$

Henceforth $\mathbf{d}_\varepsilon$ is assumed to be equal to zero, because, first, the third term in the integrand in (7.4) is small by virtue of the smallness of the tensor $\boldsymbol{\varepsilon}$ itself and, second, for a fairly dense capillary network the surface integrals of the first two terms can be neglected by virtue of the rapid oscillations of the integrands about the zero value.

Now, to obtain the second relation in (7.2), we perform the differentiation in (7.3) using to Leibnitz' rule and take into account that the following approximate equality holds within the assumptions made

$$\dot{m}^0 \boldsymbol{\varepsilon} + \nabla m^0 \otimes \mathbf{v}^0 + \mathbf{v}^0 \otimes \nabla m^0 = 0$$

The last relation in (7.2) follows from the definition of ξ^α and the following transformations

$$\left\langle m\rho p \frac{d}{dt}\left(\frac{1}{\rho}\right) \right\rangle = \frac{m^0 \rho^0}{\rho_0^0} p^0 \frac{d\xi^0}{dt} + \left\langle m\rho p \frac{d\xi}{dt} \right\rangle_+ \cong$$

$$\cong m_0^0 \frac{d\xi^0}{dt} \langle p\theta \rangle_+ + \left\langle m_0 p \theta \frac{d\xi^0}{dt} \right\rangle_+ = \left\langle p\theta \frac{d}{dt}(m_0^0 \xi^0 + m_0 \xi) \right\rangle_+ \cong \langle p \dot{\eta} \rangle_+$$

where we have used the previously adopted approximate equality

$$p^0 \cong \hat{p} = \langle \theta p \rangle_+ \tag{7.5}$$

Note that within the assumptions made relation (7.5) can be obtained from the following simple arguments. In the initial sate, the following equality holds by definition

$$p_0^0 = p_0 = \sum_{\alpha=1}^{n} \theta_\alpha p_0^\alpha = \langle \theta p_0 \rangle_+$$

Since motions with small deviations from the initial state are considered, it may be assumed to second-order of accuracy that (7.5) is valid for all permissible motions of a porous medium.

Taking relations (7.2) into account, we can write equality (6.2) in the form

$$\rho \dot{E} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + \langle p \cdot \eta \rangle_+ + \langle m_0 \rho_0 T \dot{s}_\Delta \rangle \tag{7.6}$$

The constitutive relations can be obtained from this equality if it is assumed that the internal energy E is a function of $\boldsymbol{\varepsilon}$, η and s_Δ. In this case, taking into account the rule for differentiating a composite function and the mutual independence of $\boldsymbol{\varepsilon}$, η and s_Δ, from (7.6) we obtain the relations

$$\boldsymbol{\sigma} = \rho \frac{\partial E}{\partial \boldsymbol{\varepsilon}}, \quad p = \rho \frac{\partial E}{\partial \eta}, \quad m_0 \rho_0 T = \rho \frac{\partial E}{\partial s_\Delta} \tag{7.7}$$

It is natural to presume that they assign a one-to-one correspondence between the system of variables $\boldsymbol{\varepsilon}$, η, s_Δ and the system of variables $\boldsymbol{\sigma}$, p, T. Then, along with relations (7.7), the following inverse functions also exist

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{\sigma}, p, T), \quad \eta = \eta(\boldsymbol{\sigma}, p, T), \quad s_\Delta = s_\Delta(\boldsymbol{\sigma}, p, T) \tag{7.8}$$

To find them, we can use the thermodynamic potential $H = H(\boldsymbol{\sigma}, p, T)$, which is defined by the equality

$$\rho H(\boldsymbol{\sigma}, p, T) = \rho E(\boldsymbol{\varepsilon}, \eta, s_\Delta) - \boldsymbol{\sigma} : \boldsymbol{\varepsilon} - \langle p\eta \rangle_+ - \langle m_0 \rho_0 T s_\Delta \rangle \tag{7.9}$$

and is identical apart from the sign, to the Legendre transform of the function $E = E(\boldsymbol{\varepsilon}, \eta, s_\Delta)$.[12,13] We stress that the variables $\boldsymbol{\varepsilon}$, η and s_Δ are assumed to be related to $\boldsymbol{\sigma}$, p and T by the functions (7.8). Taking into account the definition of the Legendre transform,[13] we can at once write the relations

$$\boldsymbol{\varepsilon} = -\rho \frac{\partial H}{\partial \boldsymbol{\sigma}}, \quad \eta = -\rho \frac{\partial H}{\partial p}, \quad m_0 \rho_0 s_\Delta = -\rho \frac{\partial H}{\partial T} \tag{7.10}$$

We shall make the natural assumption for small strains that the internal energy E is a convex function of its arguments. Since the Legendre transform of a convex function is also a convex function,[12] the thermodynamic potential H will be a concave function.

We will now obtain explicit expressions of the constitutive relations for small η, s_Δ and small strains. We can confine ourselves to an expansion of the internal energy in powers of $\boldsymbol{\varepsilon}$, η and s_Δ to second-order terms. The thermodynamic potential E corresponding to such an expansion will be a quadratic form in $\boldsymbol{\sigma}$, p_Δ and T_Δ, where

$$p_\Delta = (p_\Delta^1, p_\Delta^2, \ldots, p_\Delta^n), \quad T_\Delta = (T_\Delta^1, T_\Delta^2, \ldots, T_\Delta^n), \quad p_\Delta^\alpha = p^\alpha - p_0^\alpha, \quad T_\Delta^\alpha = T^\alpha - T_0^\alpha$$

In fact, Taylor's formula for the function $E = E(\boldsymbol{\varepsilon}, \eta, s_\Delta)$ has the form (we omit the residual third-order term)

$$E = E_s s_\Delta + E_\eta \eta + E_2 \tag{7.11}$$

Here E_s and E_η are the first derivatives with respect to the variables s_Δ and η when the arguments have a zero value, and $E_2 = E_2(\boldsymbol{\varepsilon}, \eta, s_\Delta)$ is a quadratic form. Since, by definition,

$$\rho E_s = m_0\rho_0 T_0,\quad \rho E_\eta = p_0$$

expression (7.9) can be reduced into the form

$$\rho H = \rho E_2 - \langle m_0\rho_0(T - T_0)s_\Delta\rangle - \langle (p - p_0)\eta\rangle_+ - \boldsymbol{\sigma} : \boldsymbol{\varepsilon}$$

Since the constitutive relations are linear, it follows from Eq. (7.11) that $\boldsymbol{\varepsilon}$, η and s_Δ can be represented in the form of linear functions of $T - T_0, p - p_0$ and $\boldsymbol{\sigma}$. Therefore, H is a quadratic form with respect to the variables $T - T_0, p - p_0$ and $\boldsymbol{\sigma}$. We will assume that the function H can be represented in the form

$$\begin{aligned}-2\rho H = {} & (H_{\boldsymbol{\sigma\sigma}}\boldsymbol{\sigma}, \boldsymbol{\sigma}) + (H_{pp}p_\Delta, p_\Delta) + (H_{TT}T_\Delta, T_\Delta) + (H_{\boldsymbol{\sigma}p}p_\Delta, \boldsymbol{\sigma}) + \\ & + (H_{p\boldsymbol{\sigma}}\boldsymbol{\sigma}, p_\Delta) + (H_{pT}T_\Delta, p_\Delta) + (H_{Tp}p_\Delta, T_\Delta) + (H_{\boldsymbol{\sigma}T}T_\Delta, \boldsymbol{\sigma}) + (H_{T\sigma}\boldsymbol{\sigma}, T_\Delta)\end{aligned} \tag{7.12}$$

Here $H_{\boldsymbol{\sigma\sigma}}$, H_{pp}, … are linear operators. For example, the operator $H_{p\boldsymbol{\sigma}}$ specifies a linear mapping from a space of stress tensors (a space of bivalent tensors) to an n-dimensional space, and $H_{\boldsymbol{\sigma}p}$ specifies a mapping from an n-dimensional space to a space of stress tensors. In a coordinate representation, these operators may be written in the form

$$(H_{p\boldsymbol{\sigma}}\boldsymbol{\sigma})^m = \frac{\partial^2 H}{\partial p^m \partial\sigma_{ij}}\sigma_{ij};\quad (H_{\boldsymbol{\sigma}p}p)_{ij} = \frac{\partial^2 H}{\partial\sigma_{ij}\partial p^m}p^m;\quad m = 1, 2, \ldots, n;\quad i, j = 1, 2, 3$$

The corresponding bilinear forms are written as follows:

$$(H_{p\boldsymbol{\sigma}}\boldsymbol{\sigma}, p_\Delta) = (H_{\boldsymbol{\sigma}p}p_\Delta, \boldsymbol{\sigma}) = \frac{\partial^2 H}{\partial p^m \partial\sigma_{ij}}\sigma_{ij}p_\Delta^m$$

For the first equality to be valid, we must assume equality of the second mixed derivatives of H.

Before writing the explicit expansion of the thermodynamic potential in small increments of the arguments, we will formulate some physical assumptions regarding the properties of a porous medium.

1. The solid phase is isotropic and the distribution of the capillary channels and pores is also isotropic.
2. The entropy of the solid phase depends on its own phase temperature and the average values of the stress tensor in the solid phase.
3. The entropy of each fluid phase depends on its phase temperature and its phase pressure.
4. The average value of the strain tensor in the solid phase depends on the effective stresses, its phase temperature and the phase pressures of the fluid phases.
5. The dilatancy of each fluid phase depends on the respective phase temperature, the phase pressure and the effective stresses.

It follows from assumption 1 that the thermodynamic potential will depend only on the invariants of the effective stress tensor J_1, J_2, J_3, the phase temperatures T^α and the phase pressures of the fluid phases p^α: $H = H(J_1, J_2, J_3, T, p)$. The dependence on the third invariant can be neglected because the inclusion of this quantity in the expansion of the thermodynamic potential makes a contribution only to third-order terms.

In order to satisfy assumptions 1-5, we must adopt the following representations for the bilinear forms from (7.12)

$$\begin{aligned}
&(H_{\boldsymbol{\sigma\sigma}}\boldsymbol{\sigma},\boldsymbol{\sigma}) = \frac{1+\nu}{E}\left(J_2-\frac{\nu}{1+\nu}J_1^2\right), \quad (H_{pp}p_\Delta, p_\Delta) = \sum_{i=1}^{n} m^i\gamma^i(p^i_\Delta)^2 = \langle m\gamma p_\Delta p_\Delta\rangle_+ \\
&(H_{TT}T_\Delta, T_\Delta) = \sum_{i=0}^{n} m^i c^i\frac{(T^i_\Delta)^2}{T^i_0} = \left\langle mc\frac{T_\Delta T_\Delta}{T_0}\right\rangle \\
&(H_{\boldsymbol{\sigma}p}p_\Delta,\boldsymbol{\sigma}) = \frac{\partial^2 H}{\partial J_1\partial p^i}p^i_\Delta J_1 = J_1\left(\sum_{i=1}^{n} m^i\beta^i p^i_\Delta\right) = J_1\langle m\beta p_\Delta\rangle_+ \\
&(H_{\boldsymbol{\sigma}T}T_\Delta,\boldsymbol{\sigma}) = \frac{\partial^2 H}{\partial J_1\partial T^i_\Delta}T^i_\Delta J_1 = -\alpha^\sigma T^0_\Delta J_1 \\
&(H_{Tp}p_\Delta, T_\Delta) = \frac{\partial^2 H}{\partial T^i\partial p^i}p^i_\Delta T^i_\Delta = \sum_{i=0}^{n} m^i\alpha^i p^i_\Delta T^i_\Delta = \langle m\alpha p_\Delta T_\Delta\rangle
\end{aligned} \tag{7.13}$$

Here ν is Poisson's ratio, E is Young's modulus, α^σ and α^i are the volume expansion coefficients, β^i and γ^i are the compressibilities, and c^i is the product of the heat capacity and the density.

The representations for $H_{\boldsymbol{\sigma\sigma}}$ and $H_{\boldsymbol{\sigma}p}$ follow from assumption 1. Assumptions 2 and 3 lead to the representations for H_{TT}, $H_{T\boldsymbol{\sigma}}$ and H_{Tp}, and assumption 5 provides the representation for H_{pp}. Assumption 4 then holds automatically.

We now substitute expressions (7.12), taking representations (7.13) into account into equalities (7.10) and write the constitutive relations in the form

$$\begin{aligned}
&\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_\sigma+\boldsymbol{\varepsilon}_p+\boldsymbol{\varepsilon}_T;\ \boldsymbol{\varepsilon}_{\boldsymbol{\sigma}} = \frac{1+\nu}{E}\left(\boldsymbol{\varepsilon}-\frac{\nu}{1+\nu}J_1\mathbf{I}_1\right),\ \boldsymbol{\varepsilon}_p = \langle m\beta p_\Delta\rangle_+\mathbf{I}_1,\ \boldsymbol{\varepsilon}_T = -\alpha^\sigma T^0_\Delta\mathbf{I}_1 \\
&\eta^k = \eta^k_\sigma+\eta^k_p+\eta^k_T;\quad \eta^k_\sigma = m^k_0\beta^k J_1,\quad \eta^k_p = m^k_0\gamma^k p^k_\Delta,\quad \eta^k_T = m^k_0\left(\frac{m^0_0}{m_0}\alpha^0 T^0_\Delta+\alpha^k T^k_\Delta\right) \\
&m^0_0\rho^0_0 s^0_\Delta = -\alpha^\sigma J_1+m^0_0\alpha^0 p^0_\Delta+m^0_0 c^0\frac{T^0_\Delta}{T^0_0},\quad \rho^k_0 s^k_\Delta = \alpha^k p^k_\Delta+c^k\frac{T^k_\Delta}{T^k_0}
\end{aligned} \tag{7.14}$$

Here $k=1, 2, \ldots, n$, and $\mathbf{I}_1$ is a unit tensor.

The expression for $\boldsymbol{\varepsilon}_p$ can be reduced to the form

$$\begin{aligned}
&\boldsymbol{\varepsilon}_p = \langle m\beta p_\Delta\rangle_+\mathbf{I}_1 = \langle m\beta\rangle_+ p^0_\Delta\mathbf{I}_1+\frac{1}{2m_0}\sum_{l=1}^{n}\sum_{k=1}^{n} m^l_0 m^k_0(\beta^l-\beta^k)(p^l-p^k-p^{lk}_0)\mathbf{I}_1, \\
&p^{ik}_0 = p^l_0-p^k_0
\end{aligned} \tag{7.15}$$

This requires the use of the identity (2.14), in which c^l is replaced by m^l_0, b^l is replaced by β^l, a^l is replaced by p^l_Δ, and the relations $\langle m_0\rangle + = m_0$, $\langle m_0 p_\Delta\rangle + = m_0 p^0_\Delta$ are taken into account.

Running ahead, we can say that with some additional assumptions, allowance for the surface-tension forces will enable us to obtain the approximate equality

$$p^\alpha - p^\beta = p^{\alpha\beta}_c(\theta_1, \theta_2, \ldots, \theta_{n-1}) = p^{\alpha\beta}_c(\theta)$$

In this case, in (7.15) the difference $p^l - p^k$ is replaced by $p_c^{lk}(\theta)$. For example, for a three-phase porous medium, in which the number of fluid phases is equal to two, we have

$$\boldsymbol{\varepsilon}_p = \langle m\beta\rangle_+ \overset{0}{p}_\Delta \mathbf{I}_1 + \frac{\overset{1}{m}_0 \overset{2}{m}_0}{m_0}(\beta^1 - \beta^2)(p_c(\theta) - p_c(\theta_0))\mathbf{I}_1; \quad p_c(\theta) = p_c^{12}(\theta) \tag{7.16}$$

Here $\theta = \theta_l$ is the saturation of the first phase, and θ_0 is the saturation in the initial state and is defined as the solution of the equation

$$p_c(\theta_0) = \overset{12}{p}_0 = \overset{1}{p}_1 - \overset{2}{p}_0$$

In another special case, in which there is only one fluid phase, the first equality in (7.10) gives

$$\boldsymbol{\varepsilon}_p = m_0 \beta \overset{0}{p}_\Delta \mathbf{I}_1, \quad \beta = \beta_1$$

which can also be obtained from the first equality in (7.16) if we set $\overset{2}{m}_0 = 0$.

It is interesting to compare the representation of the strain tensor in the form (7.16) with the expression previously proposed for it in Ref. 2, where the strain tensor was written in the isothermal case in the form

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^f + \boldsymbol{\varepsilon}^\rho + \boldsymbol{\varepsilon}^\theta$$

These terms are given the following definitions: $\boldsymbol{\varepsilon}^f$ is the over packing strain, $\boldsymbol{\varepsilon}^\rho$ is the strain caused by the variation of the density of the material of the solid particles, and $\boldsymbol{\varepsilon}^\theta$ is the strain caused by the variation of the capillary forces. This representation is based on heuristic arguments, rather than on the hypotheses made to describe a three-phase porous medium. In particular, the quantity

$$\varepsilon^\theta = \frac{1}{3}\beta^\theta (p_c(\theta) - p_c(\theta_0))$$

which appears in Ref. 2 as a deus ex machina, is nevertheless identical, apart from the notation to the second term on the right-hand side of (7.16).

8. Small strains, convective terms and the relations between generalized thermodynamic forces and fluxes

We will obtain relations which can be used to express q^α, $\mathbf{R}^{\alpha\beta}$, $Q^{\alpha\beta}$, $\dot\theta_\alpha - \dot\theta_\beta$ in terms of ∇T^α, $\mathbf{v}^\alpha - \mathbf{v}^\beta$, $1/T^\alpha - 1/T^\beta$, $p^\alpha - p^\beta - p_c^{\alpha\beta}$. For this purpose, we use (6.3).

We will first introduce the average entropy s of a porous medium defined by the formula

$$\rho s = \langle m_0 \rho_0 s\rangle + m \overset{c}{\rho}_0 \overset{c}{s}$$

as well as new notation for quantities that appear on the right-hand side of Eq. (6.3):

$$X^\alpha = -\frac{\overset{\alpha}{m}_0 \overset{\alpha}{\mathbf{q}}}{T^\alpha}, \quad x_\alpha = \frac{\nabla T^\alpha}{T^\alpha}, \quad \mathbf{r}_{0\alpha} = -\mathbf{r}_{\alpha 0} = \frac{(\mathbf{v}^\alpha - \mathbf{v}^0)}{T^\alpha}, \quad q_{\alpha\beta} = \left(\frac{1}{T^\alpha} - \frac{1}{T^\beta}\right);$$

$$\alpha, \beta = 0, 1, \dots, n$$

$$\mathbf{r}_{\alpha\beta} = (\mathbf{v}^\beta - \mathbf{v}^\alpha)\left(\frac{1}{T^\alpha} + \frac{1}{T^\beta}\right), \quad \alpha, \beta = 1, 2, \dots, n, \quad \alpha \neq \beta$$

In accordance with the concepts of the thermodynamics of non-equilibrium processes, we shall regard the quantities X^α, $Y^{\alpha\beta}$, $\mathbf{R}^{\alpha\beta}$ and $Q^{\alpha\beta}$ as generalized thermodynamic forces, and we shall regard the quantities x_α, $y_{\alpha\beta}$, $\mathbf{r}_{\alpha\beta}$ and $q_{\alpha\beta}$ as generalized thermodynamic fluxes. We use $x = \{x_\alpha\}$, $y = \{y_{\alpha\beta}\}$, $\mathbf{r} = \{\mathbf{r}_{\alpha\beta}\}$ and $q = \{q_{\alpha\beta}\}$ to denote the sets of all the fluxes x_α, $y_{\alpha\beta}$, $\mathbf{r}_{\alpha\beta}$ and $q_{\alpha\beta}$, respectively.

Neglecting the small convective terms, we can represent the left-hand side of Eq. (6.3) in the form

$$\left\langle m\rho\frac{ds}{dt}\right\rangle + m_0\rho^c\frac{ds^c}{dt} \cong \rho\dot{s}$$

and write Eq. (6.3) in the new notation as follows

$$\begin{gathered}\rho\dot{s} = -\nabla\cdot J + X, \quad X = \mathcal{X}_q + \mathcal{X}_R + \mathcal{X}_Q + \mathcal{X}_\theta + \mathcal{X}_\tau \\ \mathcal{X}_q = \sum_{\beta=0}^{n} X^\beta x_\beta, \ \mathcal{X}_R = \sum_{\alpha=1}^{n}\sum_{\beta=[\alpha]} \mathbf{R}^{\alpha\beta}\cdot\mathbf{r}_{\alpha\beta}, \ \mathcal{X}_Q = \sum_{\alpha,\beta=0}^{n} Q^{\alpha\beta}q_{\alpha\beta}, \ \mathcal{X}_\tau = \sum_{\beta=1}^{n}\boldsymbol{\tau}^\beta : \mathbf{e}(\mathbf{v}^\beta)\end{gathered} \tag{8.1}$$

The quantity ∇J is associated with reversible heat-exchange processes and can be either positive or negative. The quantity X is governed by the irreversible phase interaction processes and should be non-negative, but it can vanish if the porous system is at rest, the temperatures of all the phases are identical, and the phase pressures are connected by the relation $p^\alpha - p^\beta - p_c^{\alpha\beta} = 0$

The main task now is to find, on the basis of physical arguments, the relations between the thermodynamic forces and fluxes under which the condition $X \geq 0$ would hold for all motions of a porous medium.

It is natural to theorize that the thermodynamic forces are uniquely defined by the thermodynamic fluxes. Then, as we know, the dissipative function $\chi = \chi(x, \mathbf{r}, q, y, \mathbf{e}(\mathbf{v}))$ can be introduced. In this case, the thermodynamic forces can be expressed in terms of the fluxes on the basis of Ziegler's principle. For convex dissipative functions, these relations have the form

$$\begin{gathered}Z^l = \lambda\frac{\partial\chi}{\partial z_l}; \quad Z^l \in (X^\alpha, \ldots, Y^{\alpha\beta}, \ldots, \mathbf{R}^{\alpha\beta}, \ldots, Q^{\alpha\beta}, \ldots, \tau^\alpha) \\ z_l \in (x_\alpha, \ldots, y_{\alpha\beta}, \ldots, \mathbf{r}_{\alpha\beta}, \ldots, q_{\alpha\beta}, \ldots, \mathbf{e}(\mathbf{v}^\alpha))\end{gathered} \tag{8.2}$$

where Z^l is a general term in the sequence of generalized fluxes indicated in the first set of parentheses, z_l is a general term in the sequence of generalized fluxes indicated in the second set of parentheses, $l = 1, \ldots, N$, and N is the number of terms in these sequences. Here

$$\lambda = \chi\left(\sum_{\alpha=0}^{n} X^\alpha x_\alpha + \sum_{\alpha=1}^{n}\sum_{\beta=0}^{n}\mathbf{R}^{\alpha\beta}\cdot\mathbf{r}_{\alpha\beta} + \sum_{\alpha,\beta=0}^{n} Q^{\alpha\beta}q_{\alpha\beta} + \sum_{\alpha,\beta=1}^{n} Y^{\alpha\beta}y_{\alpha\beta} + \langle\boldsymbol{\tau} : \mathbf{e}(\mathbf{v})\rangle\right)^{-1} \tag{8.3}$$

When summation is carried out over the two indices, it is assumed that $\alpha \neq \beta$.

As we know,[14] the friction forces in fluids may depend linearly or non-linearly on the strain rate tensor. On the right-hand side of Eq. (6.3) there are two terms that are governed by viscous friction forces in the fluids: H_R, which depends on the difference between the phase velocities and is associated with the dissipation of mechanical energy due to the interaction between different phases on the boundary, and H_τ, which depends on the strain rate tensors of each phase and appears because the porous medium is not in an equilibrium state.

We will next examine the relations between H_B and H_τ, assuming that the dissipative function depends on the phase-interaction force, the rate strain tensors, the phase-temperature gradients, etc. There are three possibilities for different relations between the orders of magnitude of H_R and H_τ.

We first assume that the strain rate tensors of the fluid phases are small, so that $|H_\tau| \gg H_R$ and H_τ may be neglected under the assumption that $H_\tau \neq 0$. In this case, for any fluid phase, the shear stresses will be equal to zero ($\boldsymbol{\tau}^\alpha \equiv 0$), and the dissipative function will not depend on the strain rate tensors. For slow (quasistatic) motions, non-linear equations, which relate the average velocities of the fluid phases to the phase pressure gradients, can then be obtained from the constitutive relations. If the dissipative function is assigned in the form of a quadratic form in the phase-interaction forces, these equations will be linear, and they can be called a generalized Darcy law.

Next let H_R and H_τ have values of the order of unity. It is then natural to assume that the dissipative function will be a quadratic form in both the phase-interaction forces and the strain rate of deformation tensors of the fluid phases. In this case, as above, we can obtain a generalized Darcy law, as well as linear relations between $\boldsymbol{\tau}^\alpha$ and $\mathbf{e}(\mathbf{v}^\alpha)$.

If the values of H_R and H_τ are large, the dissipative function should depend on all the quantities mentioned above. In this case, $\tau^\alpha \neq 0$, and the constitutive relations may be either linear or non-linear.

We shall henceforth confine ourselves to linear constitutive relations. Then the dissipative function should be a positive definite quadratic form in the variables $x_{\alpha\beta}$, $\mathbf{r}_{\alpha\beta}$, $q_{\alpha\beta}$, $y_{\alpha\beta}$ and $\mathbf{e}(\mathbf{v}^\alpha)$. For simplicity, we will assume that the viscous stresses are small, and we shall neglect them, assuming that $\chi_\tau \equiv 0$. It follows from the positive definiteness of the dissipative function that χ is a convex function, $\chi \geq 0$, and $\chi(0) = 0$. In this case, $\lambda = 1/2$.

Before writing the explicit expression for the dissipative function, we point out several properties of dissipative processes, which follow from physical arguments.

1. It is natural to assume that the dissipation of mechanical energy is associated mainly with the phase interaction on the phase boundaries and depends solely on the thermomechanical parameters on the two sides of the boundary.
2. Another assumption is associated with the equivalence of all the fluid phases, i.e., among them there is no phase that could be considered distinct by any criteria. From this it follows that the dissipative function should not depend on how the numbering of the fluid phases is chosen.
3. It can be concluded on the basis of physical arguments that the quantities X^α, $\mathbf{R}^{\alpha\beta}$ ($\alpha = 1, \ldots, n$, $\beta = 0, 1, \ldots, n$), on the one hand, and the quantities $Q^{\alpha\beta}(\alpha, \beta = 0, \ldots, n)$, $Y^{\alpha\beta}(\alpha, \beta = 0, \ldots, n)$, on the other, are governed by different physical processes. In particular, X^α and $\mathbf{R}^{\alpha\beta}$ are governed by the molecular-kinetic laws of motion of the fluids, while $Q^{\alpha\beta}$ and $Y^{\alpha\beta}$ are governed by the surface phenomena.

To obtain constitutive relations that are consistent with these properties of dissipative processes, we will specify the dissipative function in the form

$$\begin{aligned}
\chi &= \chi^{qR}(x, \mathbf{r}) + \chi^{QY}(q, y) \\
\chi^{qR}(x, \mathbf{r}) &= \sum_{\alpha=0}^{n} a_\alpha^q x_\alpha^2 + \sum_{\alpha=1}^{n} a_\alpha^n \mathbf{r}_{0\alpha}^2 + \sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}^R \mathbf{r}_{\alpha\beta}^2 + \sum_{\alpha=1}^{n} a_\alpha^{qR} \mathbf{r}_{0\alpha}(\lambda_0 x_0 - \lambda_\alpha x_\alpha) + \\
&+ \sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}^{qR} \mathbf{r}_{\alpha\beta}(\lambda_\alpha x_\alpha - \lambda_\beta x_\beta) \\
\chi^{QY}(q, y) &= \frac{1}{2}\sum_{\beta=1}^{n} n_\beta^Q (q_{0\beta}^2 + q_{\beta 0}^2) + \sum_{\alpha,\beta=1}^{n} \left(\frac{1}{\tau_Q} q_{\alpha\beta}^2 + \frac{1}{\tau_\theta} y_{\alpha\beta}^2 + \frac{2}{\tau} q_{\alpha\beta} y_{\alpha\beta}\right)
\end{aligned} \tag{8.4}$$

Here a_α^q, a_α^R, $a_{\alpha\beta}^R$, a_α^{qR}, $a_{\alpha\beta}^{qR}$, a_α^Q, τ_Q, τ_θ, τ, λ_α, λ_0 are constants. The coefficient τ has the dimension of time and can be interpreted as the relaxation time. Owing to this representation of the dissipative function, the constitutive relations will satisfy the Onsager symmetry principle (the symmetry of the kinetic coefficients).

Now, taking formulae (8.2)–(8.4) into account, we obtain the constitutive relations

$$\begin{aligned}
&X^0 = \frac{1}{2}\frac{\partial \chi}{\partial x_0} = a_0^q x_0 + \lambda_0 \sum_{\alpha=1}^{n} a_\alpha^{qR} \mathbf{r}_{0\alpha}, \quad X^\alpha = \frac{1}{2}\frac{\partial \chi}{\partial x_\alpha} = a_\alpha^q x_\alpha - \lambda_\alpha a_\alpha^{qR} \mathbf{r}_{0\alpha} + \lambda_\alpha \sum_{\beta \neq 1}^{n} a_{\alpha\beta}^{qR} \mathbf{r}_{\alpha\beta} \\
&\mathbf{R}^{0\alpha} = \frac{1}{2}\frac{\partial \chi}{\partial \mathbf{r}_{0\alpha}} = a_\alpha^{qR}(\lambda_0 x_0 - \lambda_\alpha x_\alpha) + a_\alpha^R \mathbf{r}_{0\alpha}, \; \mathbf{R}^{\alpha\beta} = \frac{1}{2}\frac{\partial \chi}{\partial \mathbf{r}_{\alpha\beta}} = a_{\alpha\beta}^{qR}(\lambda_\alpha x_\alpha - \lambda_\beta x_\beta) + a_{\alpha\beta}^R \mathbf{r}_{\alpha\beta}\,(8.5) \\
&\alpha = 1, 2, \ldots, n, \quad \beta \neq \alpha
\end{aligned} \tag{8.5}$$

The expression for $Q^{\alpha\beta}$ can be represented in the explicit form

$$\begin{aligned}
Q^{\alpha\beta} &= \tau_Q^{-1}(1/T^\alpha - 1/T^\beta) + \tau^{-1}(p^\alpha - p^\beta - p_c^{\alpha\beta}), \quad \alpha, \beta = 1, \ldots, n \\
Q^{\alpha 0} &= a_\alpha^Q(1/T^\alpha - 1/T^0), \quad \alpha = 0, \ldots, n
\end{aligned}$$

We will write the constitutive relations that follow from the formula

$$Y^{\alpha\beta} = \frac{1}{2}\frac{\partial\chi}{\partial y_{\alpha\beta}}$$

in the original variables

$$m(\dot\theta_\alpha - \dot\theta_\beta)/(2nT^c) = \tau^{-1}(1/T^\alpha - 1/T^\beta) + \tau_\theta^{-1}(p^\alpha - p^\beta - p_c^{\alpha\beta}) \tag{8.6}$$

$$\dot\theta_1 + \dot\theta_2 + \ldots + \dot\theta_n = 0 \tag{8.7}$$

System (8.6), (8.7) consists of $n(n-1)/2$ differential equations (for $\alpha < \beta$). If $n \geq 3$, this system will be over defined. In order for it to be compatible, several constraints must be imposed on the $p_c^{\alpha\beta}$: the rank of the extended matrix of coefficients at the $\dot\theta_\alpha$ ($\alpha = 1, 2, \ldots$), n must be identical to the rank of the original matrix.[15] If N is the rank of the matrix of coefficients of the system, this condition is known to be equivalent to the fact that all minors of order $N+1$ of the extended matrix should vanish. It is not difficult to see that these minors can be written in the form of several linear forms with integer coefficients in $p_c^{\alpha\beta}$. Thus, the right-hand sides in system (8.6) cannot be arbitrary and must satisfy some system of linear relations with integer coefficients.

Such a conclusion can also be reached without invoking the general theory of systems of linear equations,[15] if the following simple property of system (8.6) is taken into account. The left-hand side of any equation in system (8.6) for $\alpha < \beta$ can be represented in the form

$$\dot\theta_\alpha - \dot\theta_\beta = (\dot\theta_\alpha - \dot\theta_{\alpha+1}) - (\dot\theta_{\alpha+1} - \dot\theta_{\alpha+2}) + \ldots + (\dot\theta_{\beta-1} - \dot\theta_\beta)$$

Now, substituting the right-hand sides of the corresponding equations for $(\dot\theta_\gamma - \dot\theta_{\gamma+1})$, $\alpha \leq \gamma < \beta$, we find that $p_c^{\alpha\beta}$ should satisfy the relation

$$p_c^{\alpha\beta} = p_c^{\alpha(\alpha+1)} + p_c^{(\alpha+1)(\alpha+2)} + \ldots + p_c^{(\beta-1)\beta}, \quad \alpha, \beta = 1, 2, \ldots, n; \quad \alpha \neq \beta \tag{8.8}$$

which has a clear physical interpretation.

Let us clarify this using the example of a four-phase medium, in which the pore space is filled with three fluids. We take any cross section of an arbitrary capillary. It is divided into three regions, each of which is occupied by the respective fluid. For simplicity, we will assume that each of these regions is simply connected. We will use Γ_α to denote the boundaries between them, and use Δp_α ($\alpha = 1, 2, 3$) to denote the pressure jumps on these boundaries, respectively. We will assume that the pressure in each phase varies weakly within the occupied region. In this case, the approximate expression

$$\Delta p_1 + \Delta p_2 + \Delta p_3 \cong 0$$

is clearly valid and provides a basis to theorize that the differences between the average phase pressures, which are assumed to be equal to $p_c^{\alpha\beta}$ in an equilibrium state, should satisfy similar relations. As a result, we arrive at relations (8.8).

Thus, we shall assume that conditions (8.8) are satisfied. Then, the system of equations (8.6) is equivalent to the following system of $n-1$ linear equations

$$\frac{m}{2nT^c}(\dot\theta_\gamma - \dot\theta_{\gamma+1}) = \tau^{-1}(1/T^\gamma - 1/T^{\gamma+1}) + \tau_\theta^{-1}(p^\gamma - p^{\gamma+1} - p_c^{\gamma(\gamma+1)}), \ \gamma = 1, \ldots, n-1 \ (8.9) \tag{8.9}$$

The system consisting of (8.7) and (8.9) can be solved for $\dot\theta_\alpha$, and a system of differential equations in normal form obtained.

Let us consider the example of a three-phase medium with $n = 2$. In this case, the system of equations reduces to the equation

$$m\dot\theta/(2nT^c) = \tau^{-1}(1/T^1 - 1/T^2) + \tau_\theta^{-1}(p^1 - p^2 - p_c^{12}); \quad \theta = 2\theta_1, \quad \theta_2 = 1 - \theta/2$$

Note that in the similar equation previously presented[2] the left-hand side did not contain a multiplier like m/T^c and that the dimensions of the left-hand and right-hand sides were not the same.

For a four-phase medium ($n=3$) the system of equations consisting of (8.6) and (8.7) will be compatible, if a condition that exactly matches the second relation in (5.4) is satisfied.

In the general case of an $(n+1)$-phase porous medium, we can point out the sufficient conditions under which relations (8.8) will hold. For this purpose, we must take $p_c^{\alpha\beta}$ in the form

$$p_c^{\alpha\beta} = g^{\alpha} - g^{\beta}, \quad \alpha \neq \beta; \quad g^{\alpha} = g^{\alpha}(\theta_1, \theta_2, \ldots, \theta_{n-1}), \quad \alpha = 1, 2, \ldots, n$$

Now we can ascertain why the coefficients at the fluxes $q_{\alpha\beta}$ and $y_{\alpha\beta}$ in the quadratic form $\chi^{QY}(q, y)$, which are equal to τ_Q^{-1} and τ_θ^{-1}, respectively, are assumed to be the same and do not depend on the subscripts α and β. If we reject this assumption, it will not be possible to establish the compatibility of system (8.6), (8.7) in the general case.

Let us consider the special case in which $\tau_\theta \ll 1$. In this case, from (8.6) we can obtain the approximate expressions

$$p^{\alpha} - p^{\beta} = p_c^{\alpha\beta}(\theta_1, \theta_2, \ldots, \theta_{n-1}) \tag{8.10}$$

Substituting them into constitutive relations (7.14), we obtain

$$\boldsymbol{\varepsilon}_p = \langle m\beta\rangle\langle\theta p_{\Delta}\rangle_{+}\mathbf{I}_1 + \frac{1}{2}\sum_{\alpha,\gamma=1}^{n}\frac{\overset{\alpha}{m_0}\overset{\gamma}{m_0}}{m_0}(\beta^{\alpha} - \beta^{\gamma})(p_c^{\alpha\gamma} - p_0^{\alpha\gamma})\mathbf{I}_1$$

For a three-phase medium the last relation takes the form

$$\boldsymbol{\varepsilon}_p = \langle m\beta\rangle\langle\theta p_{\Delta}\rangle_{+}\mathbf{I}_1 + \frac{\overset{1}{m_0}\overset{2}{m_0}}{m_0}(\beta^1 - \beta^2)(p_c^{12}(\theta) - p_c^{12}(\theta_0))\mathbf{I}_1 \tag{8.11}$$

Here θ_0 is the root of the equation $p_c^{12}(\theta_0) = p_0^1 - p_0^2 = p_0^{12}$.

Comparing relations (7.16) and (8.11), we can see that they are identical.

We now obtain the generalized Darcy law for a porous medium containing $n+1$ phases in the isothermal case by setting $x_0 = x_\alpha = 0$ and $T^0 = T^\alpha = T$. For this purpose, we should confine ourselves to quasistatic motions of the porous medium and neglect the inertial terms in the momentum conservation equations. Then, from system (2.6) we obtain the system of equations

$$-m^{\alpha}\nabla p^{\alpha} + \mathbf{R}^{\alpha} = 0, \quad \alpha = 1, 2, \ldots, n \tag{8.12}$$

The total phase-interaction force $\mathbf{R}^\alpha$ associated with the α phase is represented, according to (4.1) and (8.5), in the form

$$\mathbf{R}^{\alpha} = \sum_{\beta\in[\alpha]}\mathbf{R}^{\alpha\beta} = a_{\alpha}^{R}\mathbf{r}_{\alpha 0} + \sum_{\beta\in(\alpha)}a_{\alpha\beta}^{R}\mathbf{r}_{\alpha\beta} = \frac{1}{T}\left(a_{\alpha}^{R}(\mathbf{v}^0 - \mathbf{v}^{\alpha}) + 2\sum_{\beta\in(\alpha)}a_{\alpha\beta}^{R}(\mathbf{v}^{\beta} - \mathbf{v}^{\alpha})\right) =$$

$$= \frac{1}{T}\left(-a_{\alpha}^{T}(\mathbf{v}^{\alpha} - \mathbf{v}^0) - 2\sum_{\beta\in(\alpha)}a_{\alpha\beta}^{R}(\mathbf{v}^{\alpha} - \mathbf{v}^0) + 2\sum_{\beta\in(\alpha)}a_{\alpha\beta}^{R}(\mathbf{v}^{\beta} - \mathbf{v}^{\alpha})\right)$$

Expressing $\mathbf{R}^\alpha$ in terms of $\mathbf{v}^\beta - \mathbf{v}^0$ ($\beta = 1, \ldots, n$) and substituting the result into system (8.12), we obtain a system of linear equations in $\mathbf{z}^\beta = (\mathbf{v}^\beta - \mathbf{v}^0)$. We write this system in matrix form

$$\mathbf{Az} = \mathbf{P} \tag{8.13}$$

where $\mathbf{A} = ||A_{\alpha\beta}||$ is a symmetric matrix, whose elements are given by the formulae

$$A_{\alpha\alpha} = -a_{\alpha}^{R} - 2\sum_{\beta\in(\alpha)}^{n}a_{\alpha\beta}^{R}, \quad \alpha = 1, \ldots, n; \quad A_{\alpha\beta} = 2a_{\alpha\beta}^{R}, \quad \alpha, \beta = 1, \ldots, n, \quad \alpha \neq \beta,$$

$$\mathbf{P} = (m^1\nabla p^1, \ldots, m^n\nabla p^n), \quad \mathbf{z} = (\mathbf{z}^1, \ldots, \mathbf{z}^n)$$

Let $\mathbf{B} = \mathbf{A}^{-1}$ be the inverse matrix. The matrix $\mathbf{B}$ is clearly symmetric: $B_{\alpha\beta} = B_{\beta\alpha}$. Multiplying both sides of Eq. (8.13) by the matrix $\mathbf{B}$ from the left, we obtain

$$\mathbf{z} = \mathbf{BP}$$

or, in coordinate form,

$$\mathbf{v}^{\alpha} - \mathbf{v}^{0} = B_{\alpha\beta} m^{\beta} \nabla p^{\beta} \tag{8.14}$$

Formula (8.14) is an extension of Darcy's law to the case of a multiphase porous medium with n fluid phases.

Let us consider the special case of a three-phase medium ($n = 2$). In this case, the generalized Darcy law (8.14) takes the form

$$\mathbf{v}^{1} - \mathbf{v}^{0} = (a_2 + b) m^{1} \nabla p^{1} + b m^{2} \nabla p^{2}, \quad \mathbf{v}^{2} - \mathbf{v}^{0} = b m^{1} \nabla p^{1} + (a_1 + b) m^{2} \nabla p^{2}$$

Here

$$a_i = -a_i^R / d, \quad i = 1, 2; \quad b = 2a_{12}^R / d; \quad d = -(a_1 a_2 + b(a_1 + a_2))/T$$

We will also present the constitutive relations in the original variables for a three-phase medium. For this purpose, we introduce the notation

$$\tilde{T}^{k} = T^{0}/T^{k}, \quad T^{12} = 1/T^{1} + 1/T^{2}, \quad \Delta_k = (-1)^{k} T^{k} a_{12}^{qR} (\mathbf{v}^{1} - \mathbf{v}^{2}) T^{12}, \quad k = 1, 2$$

Then we have

$$m^{0}\mathbf{q}^{0} = -a_0^q \nabla T^{0} - \lambda_0 \sum_{k=1}^{2} a_k^{qR} (\mathbf{v}^{k} - \mathbf{v}^{0}) \tilde{T}^{k}, \ \mathbf{R}^{12} = -\mathbf{R}^{21} = a_{12}^{qR} (\lambda_1 x^{1} - \lambda_2 x^{2}) + a_{12}^{R} (\mathbf{v}^{2} - \mathbf{v}^{1}) T^{12}$$

$$m^{i}\mathbf{q}^{i} = -a_i^q \nabla T^{i} + \lambda_i (a_i^{qR} (\mathbf{v}^{i} - \mathbf{v}^{0}) + \Delta_i), \quad \mathbf{R}^{0i} = a_i^{qR} (\lambda_0 x^{0} - \lambda_i x^{i}) + a_i^{R} (\mathbf{v}^{i} - \mathbf{v}^{0}) / T^{i},$$
$$i = 1, 2$$

$$Q^{01} = a_1^Q (1/T^{0} - 1/T^{1}), \ Q^{02} = a_2^Q (1/T^{0} - 1/T^{2}), \ Q^{12} = \tau_Q^{-1} (1/T^{1} - 1/T^{2}) + \tau^{-1} (p^{1} - p^{2} - p_c^{12})$$

Let us consider a two-phase medium. Constitutive relations (8.5) are written in the original variables in the form

$$-m^{0}\mathbf{q}^{0} = a_0^q \nabla T^{0} + \lambda_0 a_1^{qR} (\mathbf{v}^{1} - \mathbf{v}^{0}) \tilde{T}^{1}, \quad -m^{1}\mathbf{q}^{1} = a_1^q \nabla T^{1} - \lambda_1 a_1^{qR} (\mathbf{v}^{1} - \mathbf{v}^{0}),$$
$$Q^{01} = a_1^Q (1/T^{0} - 1/T^{1})$$

The expression from which R^{01} is determined for a two-phase medium is the same as the expression for a three-phase medium.

If $\lambda_1 \ll \lambda_0$ and $\lambda_1 \ll a_1^R$, the terms containing λ_1 may be omitted, and we can obtain the relations

$$-m^{1}\mathbf{q}^{1} = a_1^q \nabla T^{1}, \quad \mathbf{R}^{01} = a_1^{qR} \lambda_0 \frac{\nabla T^{0}}{T^{0}} + a_1^{R} \frac{\mathbf{v}^{1} - \mathbf{v}^{0}}{T^{1}}$$

Similar relations were postulated in Ref. 2 on the basis of the Onsager and Curie principles. These formulae can also be obtained as a special case from the constitutive relations for a three-phase medium if we set

$$p_0^{12} \equiv 0, \quad \mathbf{v}^{1} = \mathbf{v}^{2}, \quad T^{1} = T^{2}, \quad p^{1} = p^{2}, \quad \lambda_1 = \lambda_2$$

9. The complete system of equations

Constitutive relations (7.14) enable us to express all the quantities appearing in the conservation laws in terms of the set of the following variables

$$\mathbf{v} = (\mathbf{v}^0, \mathbf{v}^1, \dots, \mathbf{v}^n), \quad \boldsymbol{\sigma}, p = (p_\Delta^1, \dots, p_\Delta^n), \quad T = (T_\Delta^0, T_\Delta^1, \dots, T_\Delta^n), \quad \theta = (\theta_1, \dots, \theta_n)$$

Now, using relations (7.14), we obtain the complete system of equations that the variables $\mathbf{v}$, $\boldsymbol{\sigma}$, p and T must satisfy. For this purpose, each of the relations (7.14) must be differentiated with respect to time. Then, the right-hand sides of the relations obtained will be linear forms in the derivatives. We replace the derivatives with respect to time $\dot{\boldsymbol{\varepsilon}}$ and $\dot{s}$ on the left-hand sides of these relations by the right-hand sides of the second equality in (7.2) and the last equality in (2.6), respectively. We replace the derivative $\dot{\eta}^\alpha$ by the right-hand side of the relation

$$\dot{\eta}^\alpha = \nabla \cdot \mathbf{v}_M^\alpha + m_0 \dot{\theta}_\alpha$$

Here $\mathbf{v}_M^\alpha = \theta_\alpha (m_0^0 \mathbf{v}^0 + m_0 \mathbf{v}^\alpha)$, $\alpha = 1, 2, \dots n$.

The expression for $\dot{\eta}^\alpha$ follows from the definitions of η^α and ξ^α and the equation of continuity, which we can write using ξ in the form (we omit the superscript)

$$\frac{m\rho}{\rho_0}\frac{d\xi}{dt} = \frac{dm}{dt} + m\nabla \cdot \mathbf{v}$$

We will introduce notation which enables as to write each equation of system (9.1) (see below) in the same form for all phases. Henceforth Latin letters are employed for the indices that are used to number the phases. We have

$$\boldsymbol{\Pi} = (\Pi^0, \Pi^1, \dots, \Pi^n), \quad \Pi^0 = \boldsymbol{\sigma}, \quad \Pi^1 = 0$$

$$\mathbf{r} = (\mathbf{r}^0, \mathbf{r}^1, \dots, \mathbf{r}^n), \quad \mathbf{r}^0 = T^0 \lambda_0 \sum_{k=1}^{n} a_k^{qR} \mathbf{r}_{0k}, \quad \mathbf{r}^l = T^l \lambda_l \left(-a_{lk}^{qR} \mathbf{r}_{0l} + \sum_{k=1}^{n} a_{lk}^{qR} \mathbf{r}_{lk} \right)$$

$$\Theta = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$T_M^l = \theta_l (m_0^0 \alpha^0 T_\Delta^0 + m_0 \alpha^l T_\Delta^l), \ \varkappa = (\varkappa^0, \dots, \varkappa^n), \ \varkappa^0 = -\alpha^\sigma, \ \varkappa^l = 0, \ l = 1, 2, \dots, n$$

$$b = (b^0, \dots, b^n), \quad b^k = a_k^q, \quad k = 0, \dots, n$$

Using this notation, we can write the system of differential equations in $\mathbf{v}$, $\boldsymbol{\sigma}$, p and T in the form

$$\begin{aligned}
& m\rho \frac{\partial \mathbf{v}}{\partial t} = \nabla \cdot \boldsymbol{\Pi} + m\nabla p_\Delta + \mathbf{R}, \\
& \frac{1+\nu}{E}\left(\frac{\partial \boldsymbol{\sigma}}{\partial t} - \frac{\nu}{1+\nu}\frac{\partial \Theta}{\partial t}\mathbf{I}_1 \right) + \left\langle m\beta \frac{\partial p_\Delta}{\partial t} \right\rangle \mathbf{I}_1 + \frac{c^0}{T_0^0}\frac{\partial T_\Delta^0}{\partial t}\mathbf{I}_1 = \mathbf{e}(\mathbf{v}^0) \\
& m\beta \frac{\partial \Theta}{\partial t} + m\gamma \frac{\partial p_\Delta}{\partial t} + \frac{\partial T_M}{\partial t} = \nabla \cdot \mathbf{v}_M + m_0 \dot{\theta} \\
& \varkappa \frac{\partial \Theta}{\partial t} + m\alpha \frac{\partial p_\Delta}{\partial t} + \frac{mc}{T_0}\frac{\partial T_\Delta}{\partial t} = \frac{\nabla \cdot (b\nabla T + \mathbf{r}) + Q}{T_0} + \Pi_S
\end{aligned} \tag{9.1}$$

The system of equations (9.1) should be supplemented by the system of differential equations (8.6), (8.7) in θ. Note that in the general case, the complete system of equations does not split into two subsystems.

Note that the last relation in system (9.1) represents a system of parabolic differential equations in the temperatures $T^0, T^1, \dots, T^n$ with Θ appearing only in the equation corresponding to the solid phase.

Remark. It was assumed above that the behaviour of the solid phase can be described by the equations of the linear theory of elasticity. However, in applications, particularly in geomechanics, media, whose behaviour is described by a model of a reinforced elastoplastic body (the flow of soft and non-cohesive soils),[16,17] or by a model of a viscoelastic body (the dilatancy of solid rocks),[18] are studied. A large part of the arguments presented here can clearly be transferred to such media. However, this will require the introduction of several additional assumptions, primarily because there is no one-to-one correspondence between the stress and strain tensors for inelastic media.

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